

# TRIGONOMETRY SUCCESS IN 20 MINUTES A DAY



- \* Answer Keys for Pre Test are included.
- \* Answer Keys for Unit work can be found in the back of the packet.
- \* Post Test May be used for a grade.



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# Introduction



If you have never taken a trigonometry course and now find that you need to know trigonometry—this is the book for you. If you have already taken a trigonometry course but felt like you never understood what the teacher was trying to tell you—this book can teach you what you need to know. If it has been a while since you have taken a trigonometry course and you need to refresh your skills—this book will review the basics and reteach you the skills you may have forgotten. Whatever your reason for needing to know trigonometry, *Trigonometry Success in 20 Minutes a Day* will teach you what you need to know.

## ► Overcoming Math Anxiety

Do you like math or do you find math an unpleasant experience? It is human nature for people to like what they are good at. Generally, people who dislike math have not had much success with math.

If you have struggles with math, ask yourself why. Was it because the class went too fast? Did you have a chance to fully understand a concept before you went on to a new one? One of the comments students frequently make is, “I was just starting to understand, and then the teacher went on to something new.” That is why *Trigonometry Success* is self-paced. You work at your own pace. You go on to a new concept only when you are ready.

When you study the lessons in this book, the only person you have to answer to is you. You don’t have to pretend you know something when you don’t truly understand. You get to take the time you need to understand everything before you go on to the next lesson. You have truly learned something only when you thoroughly understand it. Take as much time as you need to understand examples. Check your work with the answers, and if you don’t feel confident that you fully understand the lesson, do it again. You might think you don’t want to

take the time to go back over something again; however, making sure you understand a lesson completely may save you time in the future lessons. Rework problems you missed to make sure you don't make the same mistakes again.

## ► How to Use This Book

*Trigonometry Success* teaches basic trigonometry concepts in 20 self-paced lessons. The book includes a pretest; a posttest; 20 lessons, each covering a new topic; and a glossary. Before you begin Lesson 1, take the pretest. The pretest will assess your current trigonometry abilities. You will find the answer key at the end of the book. Each answer includes the lesson number that the problem is testing. This will be helpful in determining your strengths and weaknesses. After taking the pretest, move on to Lesson 1, *Angles*.

For a concise overview of the basics of trigonometry, browse Chapter 12.

Each lesson offers detailed explanations of a new concept. There are numerous examples with step-by-step solutions. As you proceed through a lesson, you will find tips and shortcuts that will help you learn a concept. Each new concept is followed by a practice set of problems. The answers to the practice problems are in an answer key located at the end of the book.

When you have completed all 20 lessons, take the posttest. The posttest has the same format as the pretest, but the questions are different. Compare the results of the posttest with the results of the pretest you took before you began Lesson 1. What are your strengths? Do you have weak areas? Do you need to spend more time on some concepts, or are you ready to go to the next level?

## ► Make a Commitment

Success does not come without effort. If you truly want to be successful, make a commitment to spend the time you need to improve your trigonometry skills.

So sharpen your pencil and get ready to begin the pretest!



# Pretest



**B**efore you begin Lesson 1, you may want to get an idea of what you know and what you need to learn. The pretest will answer some of these questions for you. The pretest is 50 multiple-choice questions covering the topics in this book. While 50 questions can't cover every concept, skill, or shortcut taught in this book, your performance on the pretest will give you a good indication of your strengths and weaknesses.

If you score high on the pretest, you have a good foundation and should be able to work your way through the book quickly. If you score low on the pretest, don't despair. This book will take you through the trigonometry concepts, step by step. If you get a low score, you may need to take more than 20 minutes a day to work through a lesson. However, this is a self-paced program, so you can spend as much time on a lesson as you need. You decide when you fully comprehend the lesson and are ready to go on to the next one.

Take as much time as you need to do the pretest. When you are finished, check your answers with the answer key at the end of the pretest. Along with each answer is a number that tells you which lesson of this book teaches you about the trigonometry skills needed for that question. You will find that the level of difficulty increases as you work your way through the pretest.



## ► Answer Sheet

1. (a) (b) (c) (d)
2. (a) (b) (c) (d)
3. (a) (b) (c) (d)
4. (a) (b) (c) (d)
5. (a) (b) (c) (d)
6. (a) (b) (c) (d)
7. (a) (b) (c) (d)
8. (a) (b) (c) (d)
9. (a) (b) (c) (d)
10. (a) (b) (c) (d)
11. (a) (b) (c) (d)
12. (a) (b) (c) (d)
13. (a) (b) (c) (d)
14. (a) (b) (c) (d)
15. (a) (b) (c) (d)
16. (a) (b) (c) (d)
17. (a) (b) (c) (d)

18. (a) (b) (c) (d)
19. (a) (b) (c) (d)
20. (a) (b) (c) (d)
21. (a) (b) (c) (d)
22. (a) (b) (c) (d)
23. (a) (b) (c) (d)
24. (a) (b) (c) (d)
25. (a) (b) (c) (d)
26. (a) (b) (c) (d)
27. (a) (b) (c) (d)
28. (a) (b) (c) (d)
29. (a) (b) (c) (d)
30. (a) (b) (c) (d)
31. (a) (b) (c) (d)
32. (a) (b) (c) (d)
33. (a) (b) (c) (d)
34. (a) (b) (c) (d)

35. (a) (b) (c) (d)
36. (a) (b) (c) (d)
37. (a) (b) (c) (d)
38. (a) (b) (c) (d)
39. (a) (b) (c) (d)
40. (a) (b) (c) (d)
41. (a) (b) (c) (d)
42. (a) (b) (c) (d)
43. (a) (b) (c) (d)
44. (a) (b) (c) (d)
45. (a) (b) (c) (d)
46. (a) (b) (c) (d)
47. (a) (b) (c) (d)
48. (a) (b) (c) (d)
49. (a) (b) (c) (d)
50. (a) (b) (c) (d)

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10  
9  
8  
7  
6  
5  
4  
3  
2  
1

Solve questions 1–29 without using a calculator.

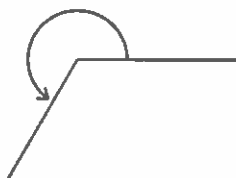
1. Convert the angle measure  $\frac{\pi}{6}$  into degrees.

a.  $30^\circ$   
b.  $60^\circ$   
c.  $180^\circ$   
d.  $45^\circ$

2. Convert  $135^\circ$  into radians.

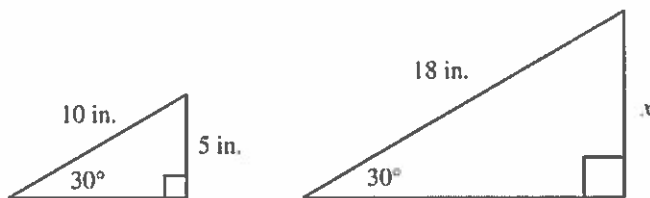
a.  $\frac{\pi}{4}$   
b.  $135\pi$   
c.  $\frac{3\pi}{8}$   
d.  $\frac{3\pi}{4}$

3. Which of the following could be the measure of this angle?



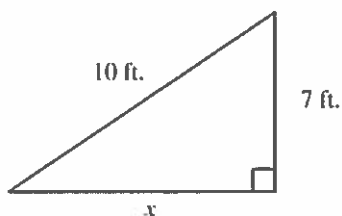
- a.  $120^\circ$   
b.  $150^\circ$   
c.  $240^\circ$   
d.  $300^\circ$
4. If a triangle has angles of measure  $53^\circ$  and  $71^\circ$ , then what is the measure of the third angle?
- a.  $56^\circ$   
b.  $62^\circ$   
c.  $66^\circ$   
d.  $124^\circ$

5. Find the length of the side marked  $x$ .



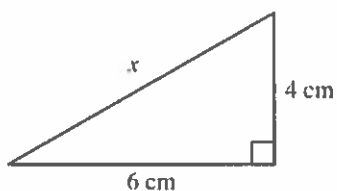
- a. 8 in.
- b. 9 in.
- c. 10 in.
- d. 13 in.

6. Find the length of the side labeled  $x$ .



- a. 3 ft.
- b.  $\sqrt{51}$  ft.
- c.  $\sqrt{61}$  ft.
- d.  $\sqrt{149}$  ft.

7. Find the length of the side labeled  $x$ .



- a. 8 cm
- b.  $2\sqrt{5}$  cm
- c.  $\sqrt{10}$  cm
- d.  $2\sqrt{13}$  cm

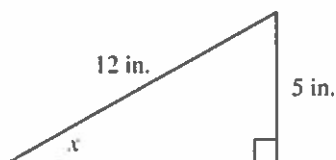
8. If  $f(x) = 4x^2 - 7x + 2$ , what is  $f(-2)$ ?

- a. 4
- b. 32
- c.  $4x^2 - 7x$
- d.  $-8x^2 + 14x - 4$

9. What is the domain of  $h(x) = \sqrt{x + 5}$ ?

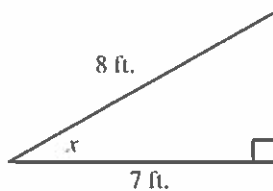
- a.  $x \geq -5$
- b.  $x < -5$
- c.  $x \geq 5$
- d.  $x > 5$

10. What is  $\sin(x)$  for the angle  $x$  in the figure?



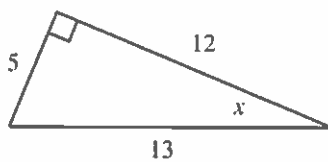
- a.  $\frac{5}{12}$
- b.  $\frac{7}{12}$
- c.  $\frac{\sqrt{119}}{12}$
- d.  $\frac{\sqrt{119}}{12}$

11. What is  $\sin(x)$  for the angle  $x$  in the figure?



- a.  $\frac{7}{8}$
- b.  $\frac{\sqrt{15}}{8}$
- c.  $\frac{\sqrt{15}}{7}$
- d.  $\frac{\sqrt{15}}{8}$

12. Find  $\cos(x)$  for the angle  $x$  in the figure.

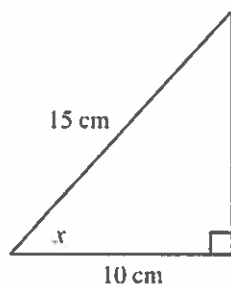


- a.  $\frac{5}{13}$
  - b.  $\frac{5}{12}$
  - c.  $\frac{12}{13}$
  - d.  $\frac{13}{12}$
13. If  $\sin(x) = \frac{3}{4}$ , then what is  $\cos(x)$ ?

- a.  $\frac{1}{4}$
- b.  $\frac{3}{5}$
- c.  $\frac{4}{3}$
- d.  $\frac{\sqrt{7}}{4}$



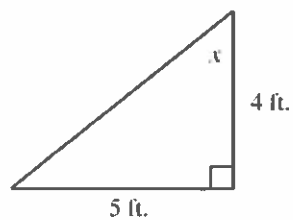
14. Find  $\tan(x)$  for angle  $x$  in the figure.



- a.  $\frac{\sqrt{5}}{2}$
  - b.  $\frac{\sqrt{5}}{3}$
  - c.  $\frac{2}{3}$
  - d.  $\frac{\sqrt{13}}{2}$
15. If  $\tan(x) = 3$ , then what is  $\sin(x)$ ?

- a. 3
- b.  $\frac{1}{3}$
- c.  $\frac{3\sqrt{10}}{10}$
- d.  $\frac{3\sqrt{2}}{4}$

Use the following figure to answer questions 16 and 17.



16. What is  $\sec(x)$  for the angle  $x$  in the figure?

a.  $\frac{\sqrt{41}}{5}$

b.  $\frac{\sqrt{41}}{4}$

c.  $\frac{3}{4}$

d.  $\frac{3}{5}$

17. What is  $\cot(x)$  for the angle  $x$  in the figure?

a.  $\frac{3}{4}$

b.  $\frac{3}{5}$

c.  $\frac{4}{5}$

d.  $\frac{5}{4}$

18. Which of the following is NOT true?

a.  $\sin^2(x) + \cos^2(x) = 1$

b.  $\frac{1}{\cot(x)} = \tan(x)$

c.  $\sec(x) = \csc(90^\circ - x)$

d.  $\csc^2(x) + 1 = \cot^2(x)$

19. Write  $\frac{\sec^2(x)}{\cot(x)}$  entirely in terms of  $\sin(x)$  and  $\cos(x)$ .

a.  $\frac{\sin(x)}{\cos^3(x)}$

b.  $\frac{\sin(x)}{\cos(x)}$

c.  $\frac{1}{\sin(x)\cos(x)}$

d.  $\frac{\cos(x)}{\sin(x)}$

20. Evaluate  $\tan(60^\circ)$ .

- a.  $\frac{1}{2}$
- b.  $\sqrt{3}$
- c.  $\frac{\sqrt{2}}{2}$
- d. 1

21. Evaluate  $\sin\left(\frac{\pi}{4}\right)$ .

- a.  $\frac{1}{2}$
- b.  $\frac{\sqrt{2}}{2}$
- c.  $\frac{\sqrt{3}}{2}$
- d. 1

22. Evaluate  $\sec(30^\circ)$ .

- a. 2
- b.  $\frac{\sqrt{2}}{2}$
- c.  $\frac{\sqrt{3}}{2}$
- d.  $\frac{2\sqrt{3}}{3}$

23. At what point is the unit circle intersected by an angle of  $315^\circ$ ? (Suppose that the angle is measured counterclockwise from the positive  $x$ -axis.)

- a.  $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
- b.  $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
- c.  $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
- d.  $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

24. What is  $\sin(240^\circ)$ ?

- a.  $\frac{1}{2}$
- b.  $-\frac{1}{2}$
- c.  $\frac{\sqrt{3}}{2}$
- d.  $-\frac{\sqrt{3}}{2}$

25. What is  $\cos\left(\frac{2\pi}{3}\right)$ ?

- a.  $\frac{1}{2}$
- b.  $-\frac{1}{2}$
- c.  $\frac{\sqrt{3}}{2}$
- d.  $-\frac{\sqrt{3}}{2}$

26. Evaluate  $\tan\left(\frac{7\pi}{4}\right)$ .

- a. 1
- b. -1
- c.  $\sqrt{3}$
- d.  $-\sqrt{3}$

27. Evaluate  $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ .

- a.  $30^\circ$
- b.  $45^\circ$
- c.  $60^\circ$
- d.  $90^\circ$

28. Evaluate  $\sin^{-1}(1)$ .

- a.  $30^\circ$
- b.  $45^\circ$
- c.  $60^\circ$
- d.  $90^\circ$

29. What is the area of an equilateral triangle with all sides of length 8 inches?

- a.  $24 \text{ in.}^2$
- b.  $32 \text{ in.}^2$
- c.  $16\sqrt{2} \text{ in.}^2$
- d.  $16\sqrt{3} \text{ in.}^2$

Use a calculator to solve questions 30–50.

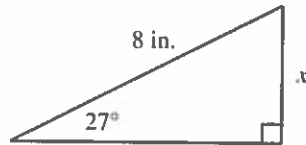
30. Suppose a 20-foot ladder is leaned against a wall from 6 feet away. Approximately how high up will it reach?

- a. 14 ft.
- b. 18 ft.
- c. 19.1 ft.
- d. 20.9 ft.

31. If  $\sin(65^\circ) = 0.9063$ , then what is  $\cos(25^\circ)$ ?

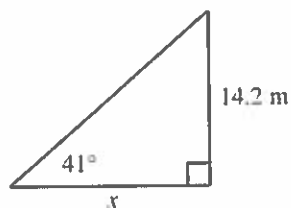
- a. .0937
- b. 0.3486
- c. 0.4244
- d. 0.9063

32. Estimate the length  $x$  in the figure.

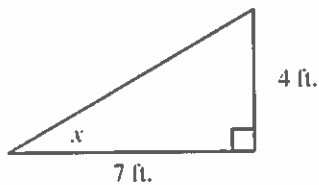


- a. 0.057 in.
- b. 3.63 in.
- c. 7.13 in.
- d. 7.65 in.

33. Estimate the length  $x$  in the figure.

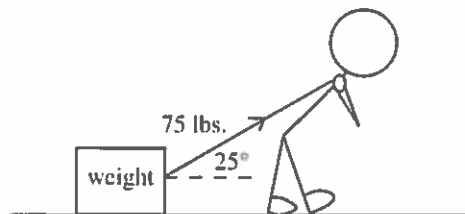


- a. 2.28 m
  - b. 16.34 m
  - c. 12.34 m
  - d. 88.39 m
34. A wire will be attached straight from the top of a 20-foot pole to the ground. If it needs to make a  $60^\circ$  angle with the ground, how long must the wire be?
- a. 10 ft.
  - b. 17.3 ft.
  - c. 23.1 ft.
  - d. 40 ft.
35. A person at the top of a 50-foot-tall tower sees a car at  $10^\circ$  below the horizontal. How far is the car from the base of the tower?
- a. 9 ft.
  - b. 51 ft.
  - c. 284 ft.
  - d. 288 ft.
36. Estimate the angle  $x$  in the figure.



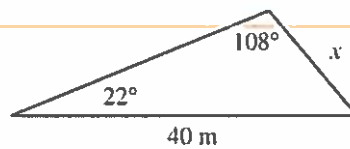
- a.  $29.7^\circ$
- b.  $34.8^\circ$
- c.  $55.2^\circ$
- d.  $60.3^\circ$

37. A 35-foot board leans against a wall. If the top of the board is 32 feet up the wall, what angle does the board make with the floor?
- a.  $24^\circ$
  - b.  $42^\circ$
  - c.  $66^\circ$
  - d.  $85^\circ$
38. A ship's destination is 30 miles north and 12 miles east. At what angle should it head to go straight to its destination?
- a.  $22^\circ$  east of north
  - b.  $24^\circ$  east of north
  - c.  $31^\circ$  east of north
  - d.  $66^\circ$  east of north
39. Suppose a mountain road descends 2,000 feet over the course of 4 miles. If the slope of the road were constant, at what angle would it descend?
- a.  $5.4^\circ$
  - b.  $6.1^\circ$
  - c.  $9.3^\circ$
  - d.  $10.1^\circ$
40. Suppose a person drags a heavy weight across the ground by pulling with 75 pounds of force at a  $25^\circ$  angle, as illustrated in the figure. How much of the force is actually pulling the weight in the direction it is moving?

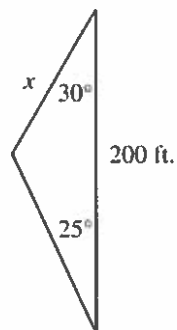


- a. 32 lbs.
- b. 35 lbs.
- c. 50 lbs.
- d. 68 lbs.

41. A group of hikers travels 2,000 meters in the direction that is  $30^\circ$  east of north. They then go 1,700 meters in the direction  $40^\circ$  north of west. At this point, in what direction would they head to go straight back to where they started?
- $5.7^\circ$  east of south
  - $6.1^\circ$  east of south
  - $12^\circ$  east of south
  - $14.2^\circ$  east of south
42. Find the length  $x$  in the figure.



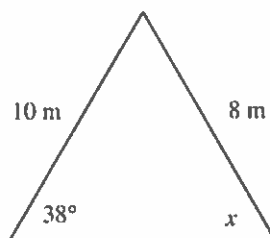
- 13.9 m
  - 15.7 m
  - 16.2 m
  - 25.4 m
43. Find the length  $x$  in the figure.



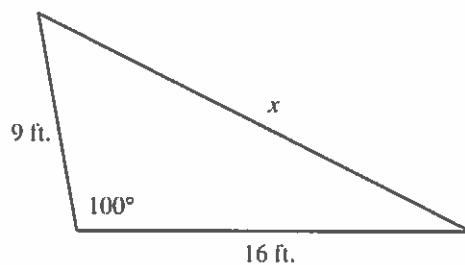
- 100 ft.
- 103 ft.
- 154 ft.
- 169 ft.



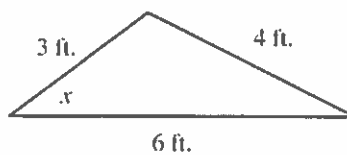
44. The following figure is not drawn accurately. Find the two possible measurements for angle  $x$ .



- a.  $50^\circ$  and  $130^\circ$
  - b.  $53^\circ$  and  $127^\circ$
  - c.  $55^\circ$  and  $125^\circ$
  - d.  $57^\circ$  and  $123^\circ$
45. Find the length  $x$  in the figure.

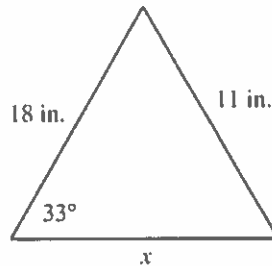


- a. 18.5 ft.
  - b. 19.7 ft.
  - c. 20.3 ft.
  - d. 20.9 ft.
46. Find the measure of angle  $x$  in the figure.



- a.  $32.1^\circ$
- b.  $36.3^\circ$
- c.  $37.5^\circ$
- d.  $41.8^\circ$

47. What are the two lengths that  $x$  could be in the figure (not drawn to scale)?



- a. 8 in. or 22 in.
  - b. 9 in. or 21 in.
  - c. 10 in. or 20 in.
  - d. 13 in. or 17 in.
48. What is the area of a triangle with lengths 5 inches, 6 inches, and 7 inches?
- a.  $15 \text{ in.}^2$
  - b.  $2\sqrt{6} \text{ in.}^2$
  - c.  $4\sqrt{6} \text{ in.}^2$
  - d.  $6\sqrt{6} \text{ in.}^2$
49. A family drives down a straight stretch of highway for 25 miles. When they begin driving, a distant mountain can be seen  $30^\circ$  off to the right of the road. When they finish driving, the mountain now appears to be  $70^\circ$  to the right of the road. How far away is the mountain from them when they have finished driving?
- a. 17 mi.
  - b. 19 mi.
  - c. 21 mi.
  - d. 23 mi.
50. A 20-foot-tall inflatable dog has been installed on top of a restaurant. The angle to the top of the dog is  $72^\circ$  from where a person on the ground is standing. The angle to the bottom of the dog is  $61^\circ$ . How far away is the restaurant from where the person is standing?
- a. 15 ft.
  - b. 20 ft.
  - c. 25 ft.
  - d. 30 ft.

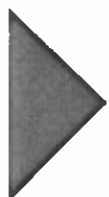
**► Answers**

1. a. Lesson 1
2. d. Lesson 1
3. c. Lesson 1
4. a. Lesson 2
5. b. Lesson 2
6. b. Lesson 3
7. d. Lesson 3
8. b. Lesson 4
9. a. Lesson 4
10. a. Lesson 5
11. d. Lesson 5
12. c. Lesson 6
13. d. Lesson 6
14. a. Lesson 7
15. c. Lesson 7
16. b. Lesson 8
17. c. Lesson 8
18. d. Lesson 8
19. a. Lesson 8
20. b. Lesson 9
21. b. Lesson 9
22. d. Lesson 9
23. a. Lesson 10
24. d. Lesson 11
25. b. Lesson 11
26. b. Lesson 11
27. a. Lesson 14
28. d. Lesson 14
29. d. Lesson 19
30. c. Lesson 3
31. d. Lesson 6
32. b. Lesson 12
33. b. Lesson 12
34. c. Lesson 13
35. c. Lesson 13
36. a. Lesson 14
37. c. Lesson 15
38. a. Lesson 15
39. a. Lesson 15
40. d. Lesson 16
41. b. Lesson 16
42. b. Lesson 17
43. d. Lesson 17
44. a. Lesson 17
45. b. Lesson 18
46. b. Lesson 18
47. c. Lesson 18
48. d. Lesson 19
49. b. Lesson 20
50. a. Lesson 20



## LESSON

# 1



# Angles

In this lesson, we examine the concept of an angle. There are two ways to measure the size of an angle: with degrees or with radians. We study both, along with the ways to convert from one to the other.

An angle ( $\angle$ ) is formed when two **line segments** (pieces of straight lines) meet at a point. In Figure 1.1,  $\angle XYZ$  consists of  $\overline{XY}$  and  $\overline{YZ}$ , which meet at point  $Y$ . Point  $Y$  is called the **vertex** of the angle.

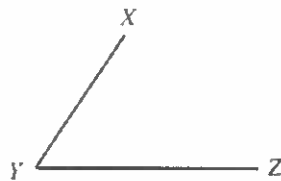


Figure 1.1

The measure of an angle is not given by the size of the lines that come together, but by how much one of the lines would have to pivot around the vertex to line up with the other. In Figure 1.2, for example,  $\overline{BC}$  would have to rotate much further around vertex  $B$  to be in line with  $\overline{AB}$  than  $\overline{EF}$  would have to rotate around  $E$  to line up with  $\overline{ED}$ . It is for this reason that the measure of  $\angle ABC$  is greater than that of  $\angle DEF$ .

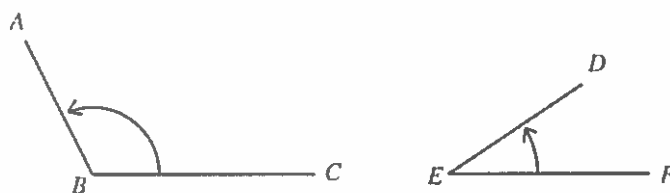


Figure 1.2

## ► Degrees

There are two ways to describe a rotation that goes all the way around, like the one in Figure 1.3. The first way is to call this **360 degrees**. The number 360 was chosen by the ancient Babylonians, who counted things in groups of 60. Modern people generally count things in groups of 10, although we still count minutes and seconds in groups of 60 because of Babylonian influences on our civilization.

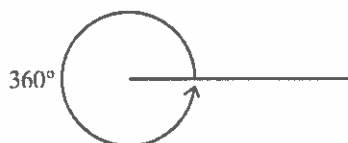


Figure 1.3

In fact, one-sixtieth of one degree is called a **minute**, and one-sixtieth of a minute is called a **second**. Astronomers and navigators sometimes still measure angles in degrees, minutes, and seconds, although mathematicians generally use decimals.

A **straight angle**, made by rotating halfway around, has a measure of 180 degrees, written  $180^\circ$ . Half of this is called a **right angle**, measuring  $90^\circ$ . Because right angles form the corners of squares and rectangles, a  $90^\circ$  angle is often indicated by putting a small square in the angle, as shown in Figure 1.4.



Figure 1.4

## ► Radians

The other way to measure a full rotation is to use the **circumference** of a circle (the distance around). The formula for the circumference ( $C$ ) of a circle is  $C = 2\pi \cdot r$ , where  $r$  is the **radius** of the circle. If we suppose the radius of the circle is one inch, one centimeter, or one of whatever units we use to measure distance, the circumference of the circle is  $C = 2\pi$  of those units, where  $\pi$  is the Greek symbol *pi*, pronounced “pie.” Such a circle is called a **unit circle** (Figure 1.5). No one has ever been able to calculate the number  $\pi$  exactly. A decent approximation is  $\pi \approx 3.14$ , although  $\pi$  is actually a tiny bit more than this.

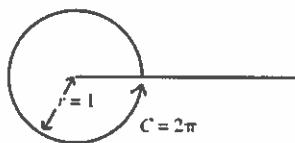


Figure 1.5

### Tip

The **squiggle equals** sign,  $\approx$ , means “is approximately equal to.” Many of the numbers that occur in trigonometry are *irrational*, which means that they cannot be written out in a finite amount of space. Thus,  $\sqrt{2} \approx 1.414$  means that when you multiply 1.414 by itself, you get a number reasonably close to 2. To get exactly 2, you would need an infinite number of decimal places, which is really asking too much! Sometimes close enough is good enough.

If we want to be perfectionists and use exact numbers, then we will have to use symbols to represent irrational numbers. Thus,  $\sqrt{2}$  is the number that multiplies by itself to get exactly 2. The exact distance around a circle with a one-unit radius is  $2\pi$ .

The **radian** measure of an angle is the distance it goes around the unit circle, or circle with a radius equal to one. For example, the radian measure of a straight angle is  $\pi$ , and the radian measure of a right angle is  $\frac{\pi}{2}$ , as shown in Figure 1.6. These measures don’t roll off the tongue like degree measures do. However, because radian measure is a distance, measured in the units of length being used, it is sometimes essential for mathematical calculations. A degree is a measure of rotation, not distance.

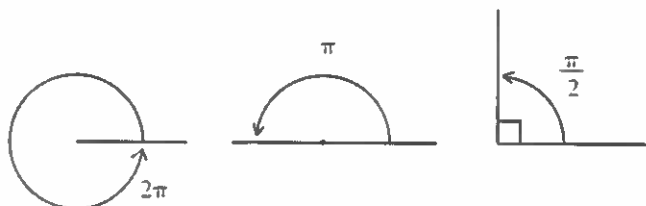


Figure 1.6

### Example

How many degrees are in one-third of a right angle? How many radians is this? A right angle consists of  $90^\circ$ . One-third of this is thus

$$\frac{1}{3} \cdot 90^\circ = 30^\circ$$

A right angle has a radian measure of  $\frac{\pi}{2}$ . One-third of this is  $\frac{\pi}{6}$  (see Figure 1.7).

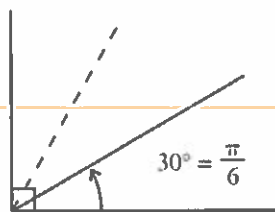


Figure 1.7

### ► Practice

For each angle, give the measure in degrees and in radians. Sketch the angle.

1. one-tenth of a full circle
2. one-third of a full circle
3. one-half of a right angle
4. two-thirds of a right angle

### ► Converting between Degrees and Radians

A full  $360^\circ$  rotation has a radian measure of  $2\pi$ . Thus, we can write that as

$$360 \text{ degrees} = 2\pi \text{ radians}$$

If we divide both sides by  $360^\circ$ , we will get  $1 = \frac{2\pi}{360^\circ}$ , which simplifies to  $\frac{\pi}{180^\circ}$ . Because multiplying by 1 doesn't change anything, we can use this to convert an angle measured in degrees to one measured in radians.



**Example**

How many radians is an angle of  $150^\circ$ ? We multiply  $150^\circ$  by  $\frac{\pi}{180^\circ}$  and get

$$150^\circ \cdot \frac{\pi}{180^\circ} = \frac{15\pi}{18} = \frac{5\pi}{6}$$

Notice that when we divide degrees by degrees, these units cancel out altogether. If we want to convert from radian measure to degrees, we multiply by  $\frac{180^\circ}{\pi}$ .

**Example**

How many degrees are in an angle with radian measure  $\frac{7\pi}{12}$ ? We multiply

$$\frac{7\pi}{12} \cdot \frac{180^\circ}{\pi} = \frac{7 \cdot 180^\circ}{12} = 7 \cdot 15^\circ = 105^\circ$$

**► Practice**

Convert the following angle measurements from degrees to radians. Sketch the resulting angle.

5.  $135^\circ$

6.  $210^\circ$

7.  $330^\circ$

8.  $315^\circ$

9.  $20^\circ$

10.  $100^\circ$

11.  $206^\circ$

12.  $144^\circ$

13.  $1^\circ$

Convert the following angles from radian measure to degrees.

14.  $\frac{5\pi}{4}$

15.  $\frac{4\pi}{3}$

16.  $\frac{5\pi}{3}$

17.  $\frac{\pi}{8}$

18.  $\frac{3\pi}{5}$

19.  $\frac{7\pi}{10}$

20.  $\frac{2}{3}$

21. 4

## LESSON

# 2



# Triangles

**T**his lesson examines triangles. We see why the angles of a triangle add up to  $180^\circ$  (or  $\pi$  radians). We also see how two triangles with the same three angles (similar triangles) will always be scaled or enlarged versions of one another.

A **triangle** consists of three sides and three angles.

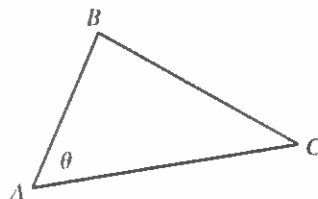


Figure 2.1

In Figure 2.1,  $\triangle ABC$  has three sides:  $\overline{AB}$ ,  $\overline{BC}$ , and  $\overline{AC}$ . The three angles could be written as  $\angle BAC$  (or  $\angle CAB$ ),  $\angle ABC$  (or  $\angle CBA$ ), and  $\angle BCA$  (or  $\angle ACB$ ). Sometimes we just write  $\angle A$  to represent  $\angle BAC$  with vertex A. Thus,  $\angle B = \angle ABC$  and  $\angle C = \angle BCA$ .

In trigonometry, we often put a variable in an angle to represent its measure. For some reason, the Greek letter  $\theta$ , pronounced **theta**, is often used to represent the measure of an angle. In Figure 2.1, the measure of  $\angle A$  is  $\theta$ .

## ► The Sum of the Angles of a Triangle

Imagine that we draw a line through point  $B$  that is parallel to  $\overline{AC}$ . This means that these lines would never intersect, no matter how far they were drawn in either direction (Figure 2.2).

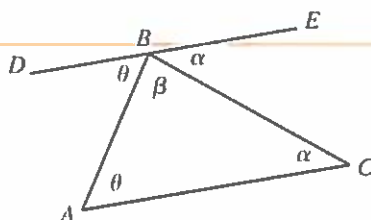


Figure 2.2

It is a result of basic geometry that the measure of  $\angle DBA$  will be the same as  $\angle A$  and the measure of  $\angle EBC$  equals the measure of  $\angle C$ . This means that the three angles of the triangle have the same measures as three angles that make up a straight angle. Thus, the three angles of any triangle add up to the measure of a straight angle, either  $180^\circ$  or  $\pi$  radians.

### Note

$$\angle A + \angle B + \angle C = 180^\circ = \pi \text{ radians}$$

### Example

If two angles of a triangle measure  $40^\circ$  and  $105^\circ$ , then what is the third angle of the triangle? If the third angle is  $\theta$ , then because the angles of a triangle add up to  $180^\circ$ :

$$\theta + 40^\circ + 105^\circ = 180^\circ$$

$$\theta = 180^\circ - 145^\circ = 35^\circ$$

Thus, the third angle measures  $35^\circ$ .

A **right triangle** is a triangle with a right angle  $\left(90^\circ, \text{ or } \frac{\pi}{2}\right)$ .

### Example

If one angle of a right triangle has a radian measure of  $\frac{\pi}{5}$ , what is the measure of the third angle? We know that the triangle has angles of  $\frac{\pi}{5}$  and  $\frac{\pi}{2}$  (the right angle). If the third angle is  $\theta$ , then the sum of the three must be  $\pi$  radians.

$$\frac{\pi}{5} + \frac{\pi}{2} + \theta = \pi$$

$$\theta = \pi - \frac{\pi}{5} - \frac{\pi}{2} = \frac{10\pi}{10} - \frac{2\pi}{10} - \frac{5\pi}{10} = \frac{10\pi - 2\pi - 5\pi}{10} = \frac{3\pi}{10}$$

Thus, the third angle measures  $\frac{3\pi}{10}$ .

### Note

If a triangle has one right angle, the other two angles must be less than  $90^\circ$ , or  $\frac{\pi}{2}$ .

### ► Practice

Given two angles of a triangle, find the measure of the third angle.

1.  $30^\circ$  and  $45^\circ$
2.  $120^\circ$  and  $14^\circ$
3.  $80^\circ$  and  $80^\circ$
4.  $\frac{\pi}{4}$  and  $\frac{\pi}{3}$
5.  $\frac{2\pi}{3}$  and  $\frac{\pi}{10}$
6.  $\frac{\pi}{6}$  and  $\frac{2\pi}{5}$

Suppose a right triangle has an angle of measure  $\theta$ , as shown in Figure 2.3. Find the measure of the third angle  $\alpha$ .

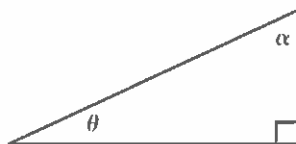


Figure 2.3

7.  $\theta = 30^\circ$

8.  $\theta = 45^\circ$

9.  $\theta = 15^\circ$

10.  $\theta = \frac{\pi}{3}$

11.  $\theta = \frac{5\pi}{12}$

12.  $\theta = \frac{3\pi}{10}$

Angles that add up to  $90^\circ$  (or  $\frac{\pi}{2}$ ) like this are called **complements** of one another.

### ► Isosceles and Equilateral Triangles

An **equilateral** triangle has all three sides of the same length. Equilateral triangles also have all three angles of the same measure. Because the angles of a triangle must add up to  $180^\circ$ , or  $\pi$ , radians, each angle of an equilateral triangle measures  $60^\circ$ , or  $\frac{\pi}{3}$ , radians. Equilateral triangles with sides 5 and 8 are shown in Figure 2.4.

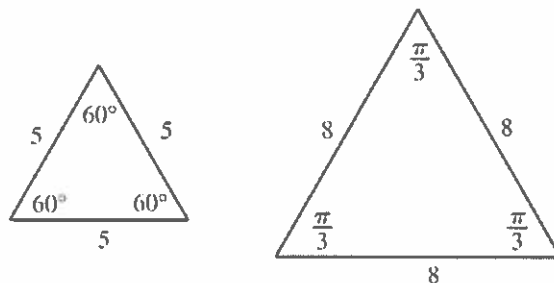


Figure 2.4

An **isosceles** triangle has two sides of the same length. The two angles that are not between these two sides must also have the same measure.

### Example

In the triangle shown in Figure 2.5, what is the measure of angle  $\theta$ ?

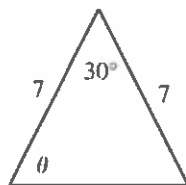


Figure 2.5

Because this triangle is isosceles, with two sides of length 7, the third angle must also have measure  $\theta$ . Because the three angles sum to  $180^\circ$ , we know that

$$\theta + \theta + 30^\circ = 180^\circ$$

$$\theta = 75^\circ$$

### ► Similar Triangles

A triangle is **scaled**, **magnified**, or **enlarged** if the length of each side is multiplied by the same number. For example, in Figure 2.6,  $\triangle DEF$  is a magnified version of  $\triangle ABC$  because the three sides of  $\triangle DEF$  are twice as big as the three sides of  $\triangle ABC$ . The **scale factor** in this example is thus 2.

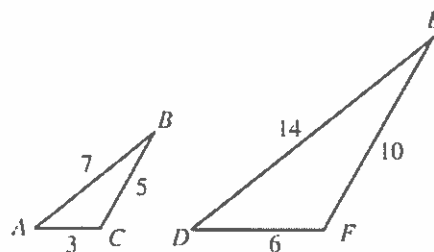


Figure 2.6

We can construct  $\triangle DEF$  with four copies of  $\triangle ABC$  as illustrated in Figure 2.7.

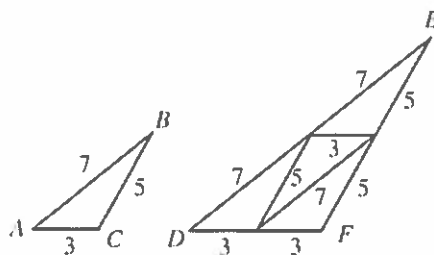


Figure 2.7

This means that the area of  $\triangle DEF$  is four times the area of  $\triangle ABC$ . It also means that the angles of  $\triangle DEF$  are the same as the angles of  $\triangle ABC$ .

If a triangle is enlarged by a scale factor of 3, then the new triangle will have sides three times longer, will have nine times the area, and will have all the same angles. The most important detail for us is that the angles remain the same. Two triangles are **similar** if they have the same three angles.

A scaled version of a triangle is always similar to the original, no matter what the scale factor may be. If each side of the second triangle is  $k$  times longer than each side in the first, then the angles are all the same. This is depicted in Figure 2.8.

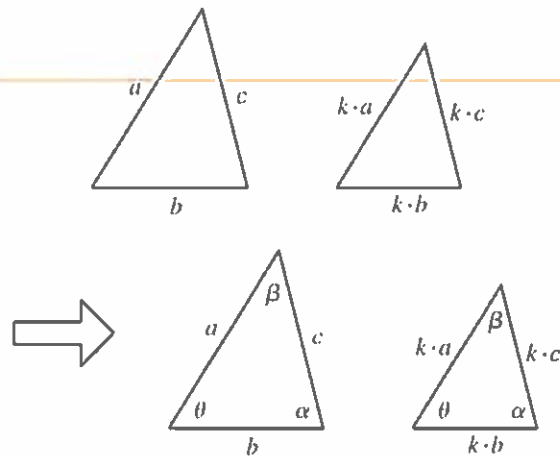


Figure 2.8

It can also be proven that if two triangles are similar, then one of the triangles is a scaled version of the other. This is illustrated in Figure 2.9. Because the triangles are similar, the sides of the second triangle must be some multiple  $k$  of the sides from the first triangle.

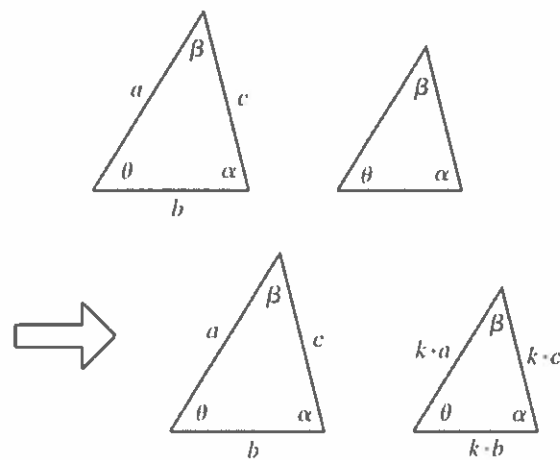


Figure 2.9



**Example**

Suppose there is a triangle with sides 8, 9, and 11 inches in length. Another triangle has the same angles  $\theta$ ,  $\alpha$ , and  $\beta$ , but the side between angles  $\alpha$  and  $\theta$  has a length of 6 inches. What are the lengths  $x$  and  $y$  of the other two sides of the triangle, as shown in Figure 2.10?

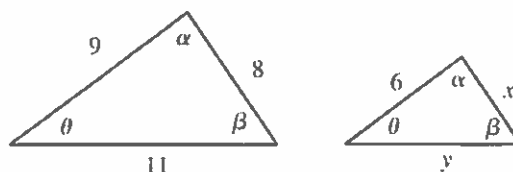


Figure 2.10

Because the two triangles have all the same angles, they are similar. Thus, the sides of the second triangle are some multiple  $k$  of the sides of the first triangle.

$$6 = k \cdot 9$$

$$x = k \cdot 8$$

$$y = k \cdot 11$$

The first equation can be solved  $k = \frac{6}{9} = \frac{2}{3}$ . Thus,

$$x = \frac{2}{3} \cdot 8 = \frac{16}{3} \text{ inches}$$

$$y = \frac{2}{3} \cdot 11 = \frac{22}{3} \text{ inches}$$

Notice that if two triangles have two angles in common, then they must have all three angles in common, because the third angle must make up the difference to  $180^\circ$ . This is shown in Figure 2.11.

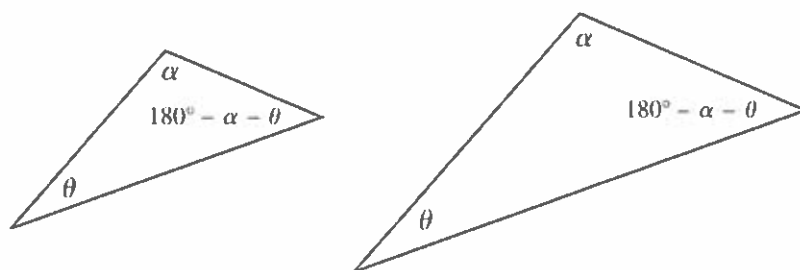


Figure 2.11

► Practice

13. One triangle has sides that are 9, 12, and 20 inches in length. Another has sides 18, 24, and 50 inches in length. Are the triangles similar?
14. One triangle has sides 4, 8, and 30 inches in length. Another has sides 6, 12, and 45 inches in length. Are the triangles similar?
15. Are these triangles similar (Figure 2.12)?



Figure 2.12

16. Are these triangles similar (Figure 2.13)?

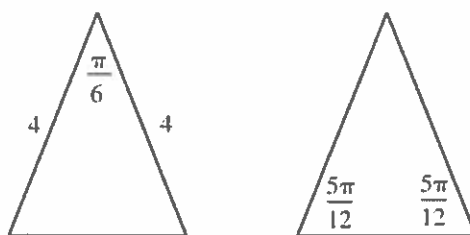
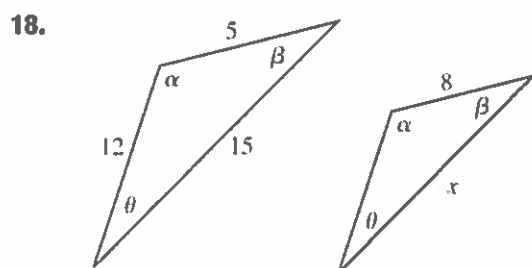
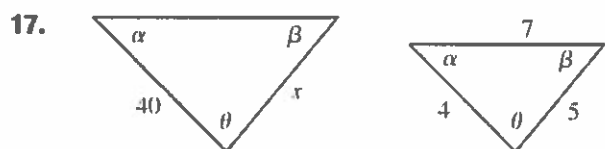


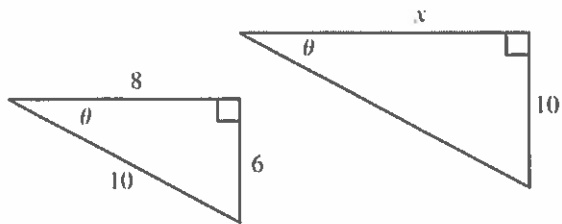
Figure 2.13

In problems 17 through 26, find the length  $x$ .

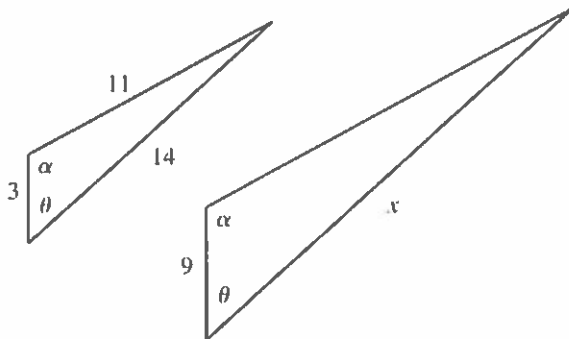


# TRIANGLES

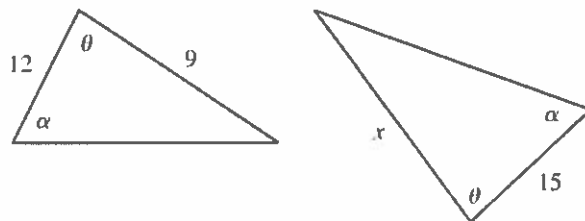
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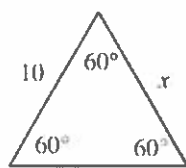
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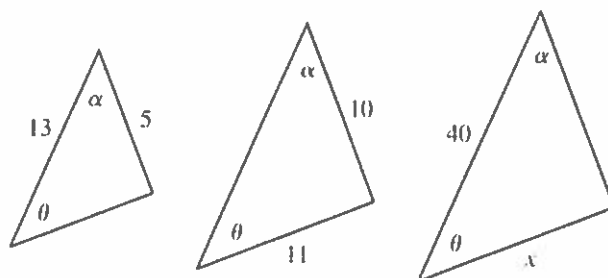
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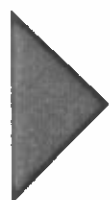
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## LESSON

# 3



# The Pythagorean Theorem

**I**n this lesson, we examine a very powerful relationship between the three lengths of a right triangle. With this relationship, we will find the exact length of any side of a right triangle, provided we know the lengths of the other two sides. We also study right triangles where all three sides have whole-number lengths.

### ► Proving the Pythagorean Theorem

The **Pythagorean theorem** is this relationship between the three sides of a right triangle. Actually, it relates the squares of the lengths of the sides. The square of any number (the number times itself) is also the area of the square with that length as its side. For example, the square in Figure 3.1 with sides of length  $a$  will have area  $a \cdot a = a^2$ .

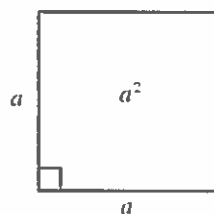


Figure 3.1

The longest side of a right triangle, the side opposite the right angle, is called the **hypotenuse**, and the other two sides are called **legs**. Suppose a right triangle has legs of length  $a$  and  $b$ , and a hypotenuse of length  $c$ , as illustrated in Figure 3.2.

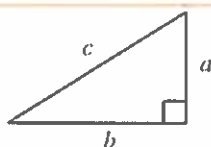


Figure 3.2

The Pythagorean theorem states that  $a^2 + b^2 = c^2$ . This means that the area of the squares on the two smaller sides add up to the area of the biggest square. This is illustrated in Figures 3.3 and 3.4.

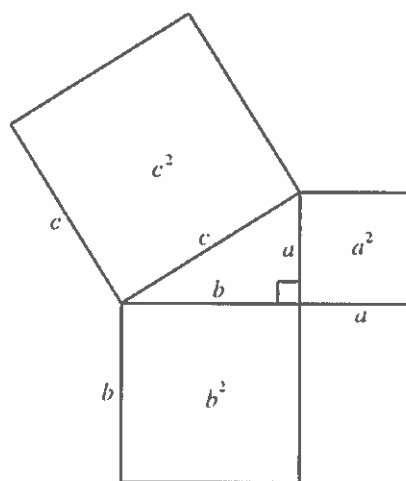


Figure 3.3

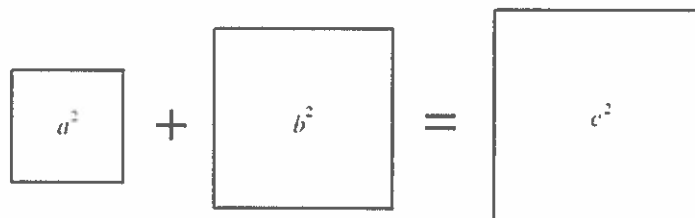


Figure 3.4

This is a surprising result. Why should these areas add up like this? Why couldn't the areas of the two smaller squares add up to a bit more or less than the big square?

We can convince ourselves that this is true by adding four copies of the original triangle to each side of the equation. The four triangles can make two rectangles, as shown in Figure 3.5. They could also make a big square with a hole in the middle, as in Figure 3.6.

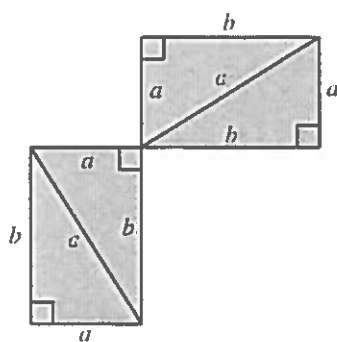


Figure 3.5

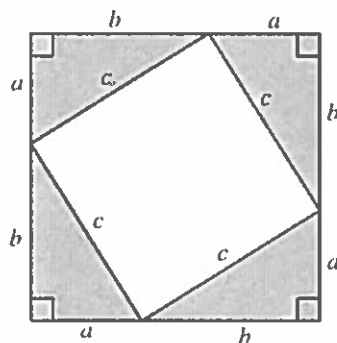


Figure 3.6

If we add the  $a^2$  and  $b^2$  squares to Figure 3.4, and the  $c^2$  square to Figure 3.5, they fit exactly. The result in either case is a big square with each side of length  $a + b$ , as shown in Figure 3.7.

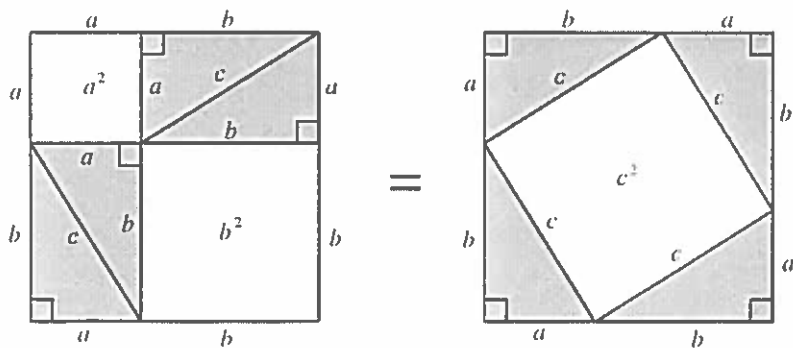


Figure 3.7

The two big squares have the same area. If we take away the four triangles from each side, we can see that the two smaller squares have the exact same area, as shown in Figure 3.8. Thus,

$$a^2 + b^2 = c^2$$

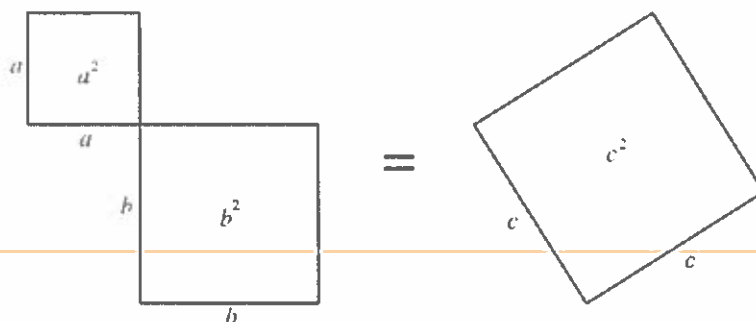


Figure 3.8

This proof of the Pythagorean theorem has been adapted from a proof developed by the Chinese about 3,000 years ago. With the Pythagorean theorem, we can use any two sides of a right triangle to find the length of the third side.

### Example

Suppose the two legs of a right triangle measure 8 inches and 12 inches, as shown in Figure 3.9. What is the length of the hypotenuse?

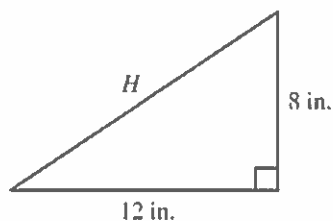


Figure 3.9

By the Pythagorean theorem:

$$12^2 + 8^2 = H^2$$

$$208 = H^2$$

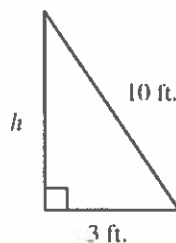
$$H = \pm\sqrt{208}$$

While the equation gives two solutions, a length must be positive, so  $H = \sqrt{208}$ . This can be simplified to  $H = 4\sqrt{13}$ .



**Example**

What is the height of the triangle in Figure 3.10?



**Figure 3.10**

Even though the height is labeled  $h$ , it is not the hypotenuse. The longest side has length 10 feet, and thus must be alone on one side of the equation.

$$h^2 + 3^2 = 10^2$$

$$h^2 = 100 - 9$$

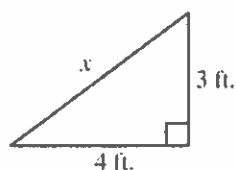
$$h = \sqrt{91} \approx 9.54$$

With the help of a calculator, we can see that the height of this triangle is about 9.54 feet.

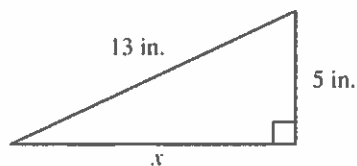
**► Practice**

Find the length of the side labeled  $x$  in each triangle.

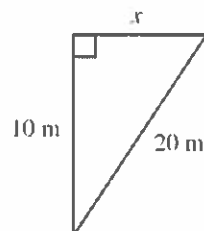
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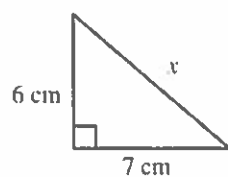
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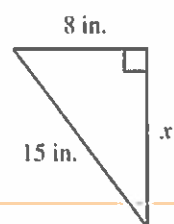
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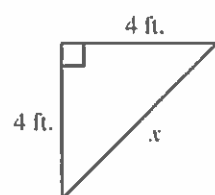
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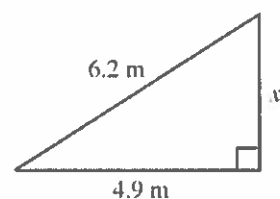
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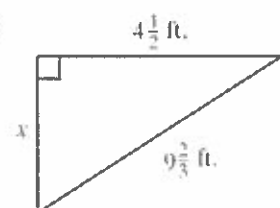
6.



7.



8.



We can use the Pythagorean theorem on triangles without illustrations. All we need to know is that the triangle is right and which side is the hypotenuse.

**Example**

If a right triangle has a hypotenuse length of 9 feet and a leg length of 5 feet, what is the length of the third side? We use the Pythagorean theorem with the hypotenuse, 9, by itself on one side, and the other two lengths, 5 and  $x$ , on the other.

$$5^2 + x^2 = 9^2$$

$$x = \sqrt{81 - 25} = \sqrt{56} = 2\sqrt{14} \approx 7.48$$

The third side is about 7.48 feet long.

**► Practice**

9. Find the hypotenuse of a right triangle with legs 2 feet and 7 feet long.
10. If a right triangle has a hypotenuse length of 10 feet and one leg length of 5 feet, what is the length of the other leg?
11. Suppose the hypotenuse of a right triangle is 50 cm long, and another side is 30 cm long. What is the length of the third side?
12. A right triangle has a one-meter leg and a two-meter leg. How long is the hypotenuse?
13. The diagonal (hypotenuse) of a large right triangle is 6 miles long. If one leg is 2 miles long, how long is the other leg?
14. What is the hypotenuse of a right triangle with both legs 6 inches in length?
15. What is the hypotenuse of a right triangle with both legs 10 inches in length?
16. Suppose a right triangle has a leg length of 1.2 meters and a hypotenuse length of 3.6 meters. What is the length of the third side?
17. The hypotenuse of a right triangle is 30 inches long. If one leg measures 9.1 inches, what is the length of the other leg?
18. What is the hypotenuse of a right triangle with legs  $3\frac{1}{2}$  feet and  $6\frac{1}{2}$  feet long?

## ► Pythagorean Word Problems

Many word problems involve finding a length of a right triangle. Identify whether each given length is a leg of the triangle or the hypotenuse. Then solve for the third length with the Pythagorean theorem.

### Example

A diagonal board is needed to brace a rectangular wall. The wall is 8 feet tall and 10 feet wide. How long is the diagonal?

Having a rectangle means that we have a right triangle, and that the Pythagorean theorem can be applied. Because we are looking for the diagonal, the 10-foot and 8-foot lengths must be the legs, as shown in Figure 3.19.

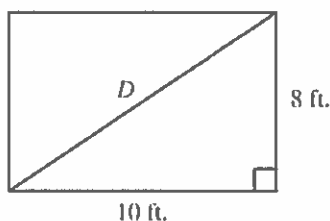


Figure 3.19

The diagonal  $D$  must satisfy the Pythagorean theorem:

$$D^2 = 10^2 + 8^2$$

$$D^2 = 100 + 64 = 164$$

$$D = \sqrt{164} = 2\sqrt{41} \approx 12.81 \text{ feet}$$

### Example

A 100-foot rope is attached to the top of a 60-foot tall pole. How far from the base of the pole will the rope reach?

We assume that the pole makes a right angle with the ground, and thus, we have the right triangle depicted in Figure 3.20. Here, the hypotenuse is 100 feet.

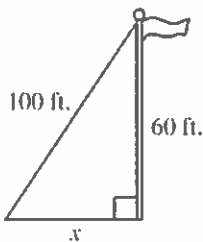


Figure 3.20

The sides must satisfy the Pythagorean theorem:

$$x^2 + 60^2 = 100^2$$

$$x = \sqrt{6,400} = 80 \text{ feet}$$

### ► Practice

19. How long is the diagonal of a standard piece of paper that measures  $8\frac{1}{2}$  by 11 inches?
20. A 20-foot ladder must always be placed 5 feet from a wall it is leaned against. What is the highest spot on a wall that the ladder can reach?
21. Televisions and computer screens are always measured on the diagonal. If a 14-inch computer screen is 12 inches wide, how tall is the screen?
22. The bottom of a cardboard box is a 20-inch square. What is the widest object that could be wedged in diagonally?

### ► Pythagorean Triples

Since the Pythagorean theorem was discovered, people have been especially fascinated by right triangles with whole-number sides. The most famous one is the 3-4-5 right triangle, but there are many others, such as 5-12-13 and 6-8-10. A **Pythagorean triple** is a set of three whole numbers  $a$ - $b$ - $c$  with  $a^2 + b^2 = c^2$ . Usually, the numbers are put in increasing order.

#### Example

Is 48-55-73 a Pythagorean triple?

We calculate  $48^2 + 55^2 = 2,304 + 3,025 = 5,329$ . It just happens that  $73^2 = 5,329$ . Thus, the numbers 48, 55, and 73 form a Pythagorean triple.

### ► Practice

Which of the following are Pythagorean triples?

23. 20-21-29

24. 15-30-34

25. 12-60-61

26. 12-35-37

27. 33-56-65

### ► Generating Pythagorean Triples

The ancient Greeks found a system for generating Pythagorean triples. First, take any two whole numbers  $r$  and  $s$ , where  $r > s$ . We will get a Pythagorean triple  $a$ - $b$ - $c$  if we set:

$$a = 2rs$$

$$b = r^2 - s^2$$

$$c = r^2 + s^2$$

This is because

$$a^2 + b^2$$

$$= (2rs)^2 + (r^2 - s^2)^2$$

$$= 4r^2s^2 + r^4 - 2r^2s^2 + s^4$$

$$= r^4 + 2r^2s^2 + s^4$$

$$= (r^2 + s^2)^2 = c^2$$

$$\text{Thus, } a^2 + b^2 = c^2.$$

### Example

Find the Pythagorean triple generated by  $r = 7$  and  $s = 2$ .

$$a = 2rs = 2(7)(2) = 28$$

$$b = r^2 - s^2 = 7^2 - 2^2 = 45$$

$$c = r^2 + s^2 = 7^2 + 2^2 = 53$$

Thus,  $r = 7$  and  $s = 2$  generate the 28-45-53 Pythagorean triple.

**► Practice**

Find the Pythagorean triple generated by the given  $r$  and  $s$ .

**28.**  $r = 4$  and  $s = 1$

**29.**  $r = 3$  and  $s = 2$

**30.**  $r = 9$  and  $s = 4$

**31.**  $r = 5$  and  $s = 3$

**32.**  $r = 10$  and  $s = 3$

If we plug in different values of  $r$  and  $s$ , we will generate many different Pythagorean triples, whole numbers  $a$ ,  $b$ , and  $c$  with  $a^2 + b^2 = c^2$ . Around 1640, a French mathematician named Pierre de Fermat suggested that this could happen only with the exponent 2. He said that no positive whole numbers  $a$ ,  $b$ , and  $c$  could possibly make  $a^3 + b^3 = c^3$  or  $a^4 + b^4 = c^4$  or  $a^n + b^n = c^n$  for any  $n > 2$ . Fermat claimed to have found a short and clever proof of this, but then died without writing it down.

For hundreds of years, mathematicians tried to prove this result, called "Fermat's Last Theorem." Only in 1995 was it finally proven, by a British mathematician named Andrew Wiles. Because his proof runs to more than 100 pages, it is clearly not the simple proof that Fermat spoke about. This assumes, of course, that Fermat had actually found a correct and simple proof.





## LESSON

# 4



# Functions

In this lesson, we study very important and useful mathematical objects called functions. If you plug a number into a function, out will come another number. This can be a handy way to relate pieces of information. We practice plugging numbers into functions. We also figure out which numbers ought not be put into certain functions. At the very end, we look at functions that are evaluated through a process instead of a formula.

### ► Defining Functions

**Functions** are a bit like equations and formulas. In fact, if you have ever seen an equation with one variable isolated on one side of the equals sign, such as

$$y = x^2 + 2$$

then you have seen a function.

Functions relate numbers to other numbers. In the equation  $y = x^2 + 2$ , when  $x = 3$ , we can compute that  $y = 3^2 + 2 = 11$ . Here, 3 relates to 11.

To evaluate other relationships, we need only plug in other numbers. What does 5 relate to? Because  $5^2 + 2 = 27$ , the number 5 relates to 27.

Every equation with two variables is going to relate numbers like this. The key to being a function is that each number should relate to only one other. If you want the number 4 to relate to 7 and 12 at the same time, then this relationship cannot be defined by a function.

For example,  $y = \pm\sqrt{25 - x^2}$  is not a function. If we want to know what the number  $x = 4$  relates to, we plug it in and get  $y = \pm\sqrt{25 - 4^2} = \pm\sqrt{9} = \pm 3$ . Here, we see that  $x = 4$  relates to two different numbers, 3 and  $-3$ . This cannot be a function.

The formula  $A = \pi r^2$  does define a function. If we plug in a number for  $r$ , only one number will come out. This formula gives the area of a circle with radius  $r$ . We could not have a circle with two different areas, so this formula is a function.

If we are going to talk about more than one function at a time, like  $y = x^2 + 2$  and  $y = \frac{x^2 + 5x}{x - 2}$ , then we give them names to identify them. Usually, we pick names that are short and easy to write, like  $f$  (for “function”) and  $g$  (for “the letter that comes after  $f$ ”). First, we write the name. Next, we represent the number to be plugged in with a variable in parentheses. Finally, we set it equal to the formula that defines the function.

$$f(x) = x^2 + 2$$

$$g(x) = \frac{x^2 + 5x}{x - 2}$$

### Tip

Parentheses are used to mean many different things in mathematics. In a function, they are used to separate the name of the function from the name of the variable that gets plugged in. This does not mean multiplication, as it did in algebra. In algebra,  $x(x + 5)$  can be multiplied out to equal  $x^2 + 5x$ . With functions,  $f(x)$  is a combination of the name of the function with the name of its variable. This has nothing to do with multiplication.

What does the function  $g$  relate to the number 7? Because the function  $g$  is written  $g(x)$ , the plug-in variable is  $x$ . Thus, we replace every  $x$  with the number 7.

$$g(7) = \frac{(7)^2 + 5 \cdot (7)}{(7) - 2} = \frac{84}{5}$$

The function  $g$  relates the number 7 with  $g(7)$ , which in this case is  $\frac{84}{5} = 16\frac{4}{5}$ .

### Example

Suppose  $s(t) = 200 + 10t - 16t^2$ . What number is related to 2?

The first part of the equation  $s(t)$  tells us that  $s$  is the name of the function and that  $t$  represents the numbers that get plugged in. The function  $s$  relates the number 2 with  $s(2)$ . We calculate this by replacing each  $t$  in the formula with the number 2.

$$s(2) = 200 + 10 \cdot (2) - 16 \cdot (2)^2 = 200 + 20 - 16 \cdot 4 = 220 - 64 = 156.$$

Thus,  $s(t) = 200 + 10t - 16t^2$  relates the number 2 to 156.

### Example

If  $cat(x) = \frac{x+1}{x^2-1}$  then what is  $cat(-3)$ ?

$$cat(-3) = \frac{(-3)+1}{(-3)^2-1} = \frac{-2}{9-1} = \frac{-2}{8} = -\frac{1}{4}$$

The function  $cat$  relates  $-3$  to the number  $-\frac{1}{4}$ .

In the last example, the name of the function,  $cat$ , had three letters. This is nothing unusual. The trigonometric functions all have three letters:  $\sin$ ,  $\cos$ ,  $\tan$ ,  $\sec$ ,  $\csc$ , and  $\cot$ .

### Tip

To help avoid errors, it is always a good idea to use parentheses around the numbers plugged into a function. For example,  $cat(-3) = \frac{(-3)+1}{(-3)^2-1}$  makes it clear that the whole  $-3$  needs to be squared in the denominator. If we had written  $cat(-3) = \frac{-3+1}{-3^2-1}$ , it would look like the denominator was  $-3^2 - 1 = -9 - 1 = -10$ , which is incorrect.

### ► Practice

1. If  $f(x) = 5x + 2$ , then to what does 3 relate under function  $f$ ?
2. If  $V(s) = s^3$ , then what is  $V(4)$ ?
3. What does the function  $g(x) = \frac{5+x}{5-x}$  relate to the number 4?

4. If  $f(x) = 10$ , what is  $f(17)$ ?
5. What is  $s(-1)$ , when  $s(t) = -16t^2 + 4t + 100$ ?
6. If  $C(r) = 2\pi r$ , then what is  $C(2.7)$ ?
7. Evaluate  $k(2)$  for  $k(u) = u^3 - 4u^2 + 2u - 5$ .
8. What is  $-1$  related to under the function  $q(x) = (x - 5)(x + 1)(x - 4)$ ?
9. If  $f(x) = \sqrt{1 - x^2}$ , then what is  $f\left(\frac{1}{2}\right)$ ?
10. When  $p(x) = \frac{x^2 + 3x - 2}{x^2 + x + 1}$ , what does  $p$  relate to the number  $-2$ ?
11. Where does  $h$  take the number  $10$  when  $h(x) = \sqrt{x^2 - 7x + 4}$ ?
12. What is  $h(-7)$ , when  $h(t) = \frac{\sqrt{2 - t}}{t + 4}$ ?
13. Evaluate  $k\left(\frac{1}{3}\right)$  for  $k(x) = \frac{x^2 + 1}{x^2 - 1}$ .
14. What does  $k(x) = \frac{x + 5}{(x - 2)(x + 7)}$  do with the number  $3$ ?

## ► Domains

Not every number can be plugged into a function. For example, if we tried to plug  $2$  into  $f(x) = \frac{5}{x - 2}$ , it would result in a division by zero:  $\frac{5}{0}$ . This is not allowed. It would be impossible, for example, to divide  $5$  pounds of nuclear waste among zero people. It might be tempting to get rid of the stuff this way, but no matter how much waste you tried to give the zero people, it simply cannot happen.

Similarly, if we tried to plug  $6$  into  $g(x) = \sqrt{2 - x}$ , then it would result in the square root of a negative number,  $\sqrt{-4}$ . What number, multiplied by itself, results in  $-4$ ? It couldn't be a negative number, because a negative times itself is always positive. Similarly, the number couldn't be either positive or zero. Thus,  $\sqrt{-4}$  is not a real number. There are places in mathematics where  $\sqrt{-1}$  is designated by the letter  $i$ , so that  $\sqrt{-4} = 2i$ . However, such numbers are not real, and thus have no place in trigonometry, which deals with the real, positive lengths of sides of triangles.

The **domain** of a function is the set of all numbers that can be plugged into it. Only the number 2 will make the denominator of  $f(x) = \frac{5}{x-2}$  become zero, and there are no square roots, so the domain of  $f$  consists of all numbers except 2. We could say that the domain was all numbers  $x$ , such that  $x < 2$  or  $x > 2$ . We could represent this in interval notation  $(-\infty, 2) \cup (2, \infty)$ , meaning that the domain consists of “all numbers greater than  $-\infty$  and less than 2, together with all numbers greater than 2 and less than  $\infty$ .” However, the most concise way to describe the domain of  $f$  is  $x \neq 2$ .

The function  $g(x) = \sqrt{2-x}$  has no denominator, so there is no danger of dividing by zero. However, we must make sure that the numbers that go into the square root are not negative. Thus, we say

$$2 - x \geq 0$$

$$-x \geq -2$$

$$x \leq 2$$

The domain of  $g$  is  $x \leq 2$ .

### Tip

Remember that multiplying or dividing both sides of an inequality by a negative number reverses the direction of the inequality.

### Example

What is the domain of  $h(x) = \frac{\sqrt{3x-2}}{x^2-9}$ ?

We must make sure that the numbers inside the square root are greater than or equal to zero, so

$$3x - 2 \geq 0$$

$$3x \geq 2$$

$$x \geq \frac{2}{3}$$

Also, we must avoid all numbers that would lead to division by zero, so

$$x^2 - 9 \neq 0$$

$$x^2 \neq 9$$

$$x \neq 3 \text{ and } x \neq -3$$

When we try to join together our two requirements,  $x \geq \frac{2}{3}$  and  $x \neq 3, -3$ , we notice that  $-3$  is not greater than  $\frac{2}{3}$ , so it is already ruled out. The domain of  $h$  is thus  $x \geq \frac{2}{3}$ , where  $x \neq 3$ . In other words, we can plug any number greater than or equal to  $\frac{2}{3}$  into the function  $h$ , except for the number 3.

### Example

What is the domain of  $f(t) = 10t^2 - 3t + 4$ ?

Here, absolutely any number can be plugged in for  $t$ , and so we say that the domain consists of all real numbers. This is often abbreviated as  $\mathbb{R}$ .

### ► Practice

Find the domain for each of the following functions.

15.  $f(x) = \frac{x+3}{x+5}$

16.  $g(x) = 3x^4 - 10x$

17.  $h(x) = \sqrt{x+3}$

18.  $y(x) = \sqrt{5x+2}$

19.  $k(t) = \frac{x+5}{x^2+6x+8}$

20.  $g(x) = \sqrt{4-x}$

21.  $f(x) = \frac{\sqrt{x+5}}{x^2-4}$

22.  $h(x) = \sqrt{3-4x}$

23.  $g(x) = x^3 - \sqrt{3-x}$

24.  $p(x) = \frac{\sqrt{2-x}}{x^3+4x^2-5x}$

25. The area  $A$  of an isosceles right triangle with base and height  $x$  is  $A(x) = \frac{1}{2}x^2$ . What is the domain of  $A$ ?

26. If a person works  $t$  hours on a project that pays \$100, then the hourly rate is  $r(t) = \frac{100}{t}$ . What is the domain of  $r$ ?

## ► Functions without Formula

Plugging numbers into a formula to evaluate the function is not so tricky. However, some functions cannot be described by a formula. Unfortunately, the trigonometric functions fall into this category. They are evaluated more by a process than by a formula. To get ready for these, we shall practice with a few functions like this.

### Example

Let  $f(x)$  = the first odd number greater than  $x$ . Evaluate  $f(4)$ ,  $f(7)$ ,  $f(2.7)$ , and  $f\left(\frac{10}{3}\right)$ .

$$f(4) = 5 \text{ because 5 is the first odd number greater than 4}$$

$$f(7) = 9 \text{ because 7 is not technically greater than 7}$$

$$f(2.7) = 3$$

$$f\left(\frac{10}{3}\right) = f\left(3\frac{1}{3}\right) = 5$$

Note that it is not difficult to evaluate  $f$  at any number, but it would be difficult to write a formula for this function.

### Example

Let  $g(x)$  = the number of letters required to spell out  $x$  in English. Evaluate  $g(3)$ ,  $g(25)$ , and  $g(7.6)$ .

$$g(3) = 5 \text{ because "three" has five letters}$$

$$g(25) = 10 \text{ because "twenty-five" has ten letters}$$

$$g(7.6) = 17 \text{ because "seven and six tenths" has 17 letters}$$

This is perhaps not the best of functions, because if you wrote "seven point six," then you would get a different answer for  $g(7.6)$ . Luckily, the trigonometric functions will all be very well defined and not subject to any confusion like this.

**► Practice**

Evaluate the following, using the functions  $h$  and  $k$  defined by:

$h(x)$  = the number of digits to the left of the decimal place in  $x$

$k(x)$  = the number  $x$  with every digit 3 replaced by 8

**27.**  $h(207.32)$

**28.**  $h(39.5)$

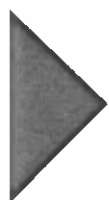
**29.**  $k(3,185)$

**30.**  $k(13.843)$



## LESSON

# 5



## Sine

In this lesson, we introduce the sine function. We show how to evaluate the sine of an angle in a right triangle. Often, this is as easy as dividing the length of one side of the triangle by another. Sometimes we need to use the Pythagorean theorem to find one of the lengths.

The first trigonometric function is **sine**. The domain of this function consists of all angles  $x$ . There is no formula for sine that can be computed directly using  $x$ . Instead, we must follow a process. For now, we will use angles between  $0^\circ$  and  $90^\circ$ . In Lesson 11, we extend the definition of sine to other angles.

Step one: Start with an angle of measure  $x$  (Figure 5.1).



Figure 5.1

Step two: Make a right triangle with angle  $x$  (Figure 5.2).

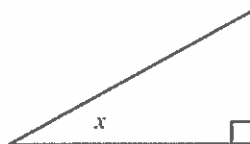


Figure 5.2

Step three: Measure the length  $H$  of the triangle's hypotenuse and the length  $O$  of the side opposite the angle  $x$  (Figure 5.3). (The **opposite** side is one that isn't part of angle  $x$ .)

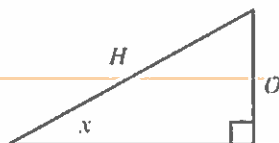


Figure 5.3

Step four: The sine of the angle  $x$  is the ratio of the two sides:

$$\sin(x) = \frac{O}{H}$$

The length of the hypotenuse  $H$  and opposite side  $O$  will depend on the size of the triangle. In step two, we could have chosen a smaller or larger right triangle with angle  $x$ , as illustrated in Figure 5.4.

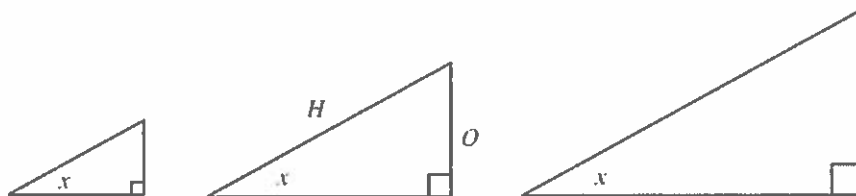


Figure 5.4

However, because the angles of a triangle add up to  $180^\circ$  (or  $\pi$  radians), the third angles of these triangles must all measure  $90^\circ - x$  degrees (or  $\frac{\pi}{2} - x$  radians). Triangles with all the same angles are similar; thus, each one is a scale multiple of the original. Suppose the smaller triangle has scale factor  $k$ , and the larger triangle has scale factor  $K$  (see Figure 5.5).

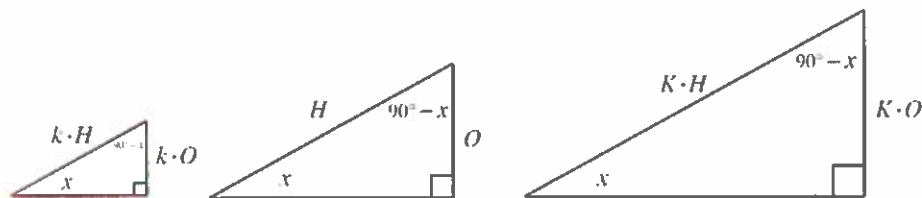


Figure 5.5

If we use the small triangle, then

$$\sin(x) = \frac{kO}{kH} = \frac{O}{H}$$

If we use the large triangle, then

$$\sin(x) = \frac{KO}{KH} = \frac{O}{H}$$

In other words, the sine of the angle does not depend on the size of the triangle used.

It is tedious to actually draw a right triangle with the correct angle  $x$ , and then to measure the lengths of the sides. There are only a few “nice angles” for which this can be done precisely, which will be discussed in Lesson 9. Calculators can be used to estimate the sine of any angle; this will be covered in Lesson 12. For the time being, it is far easier to start with a triangle and then find the sine of each of its angles.

### Example

Find the sine of the angle  $x$  in Figure 5.6.

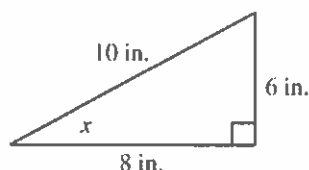


Figure 5.6

We might not know the exact measurement of angle  $x$ , in either degrees or radians, but we have everything we need to find its sine. The hypotenuse of this right triangle is  $H = 10$  inches. The side that is opposite angle  $x$  is  $O = 6$  inches. Thus,

$$\sin(x) = \frac{6 \text{ in.}}{10 \text{ in.}} = \frac{3}{5}$$

Notice that the units used to measure the lengths will always cancel out like this. From now on, we will not worry about the feet, inches, or whatever is used to measure the lengths. The numbers that come out of the sine function have no particular units. They are merely ratios.

### Example

Find the sine of angle  $\theta$  from Figure 5.7.

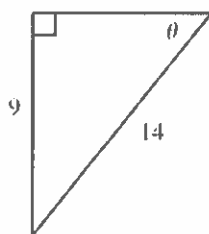


Figure 5.7

Here, the hypotenuse is  $H = 14$  and the side opposite angle  $\theta$  is  $O = 9$ , so

$$\sin(\theta) = \frac{O}{H} = \frac{9}{14}$$

### Example

Find the sine of the angle  $x$  illustrated in Figure 5.8.

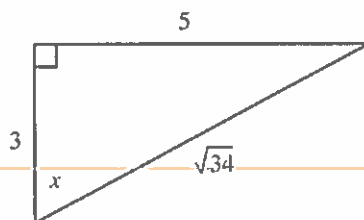


Figure 5.8

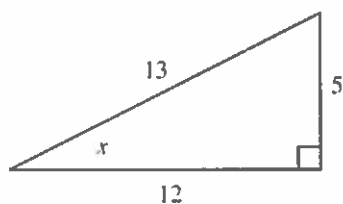
Here, the hypotenuse is  $H = \sqrt{34}$ . The opposite side (the one that doesn't touch angle  $x$ ) is  $O = 5$ . Thus,

$$\sin(x) = \frac{5}{\sqrt{34}} = \frac{5\sqrt{34}}{34}$$

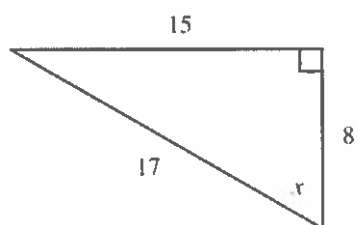
### ► Practice

Find  $\sin(x)$  for the angle  $x$  in each triangle.

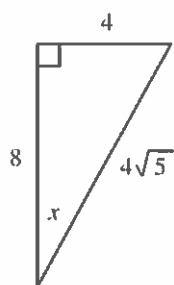
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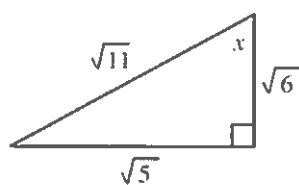
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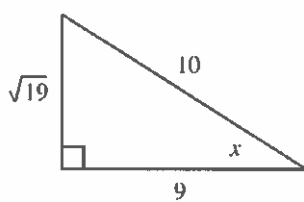
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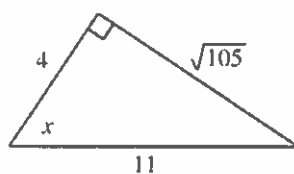
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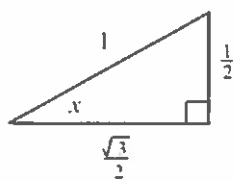
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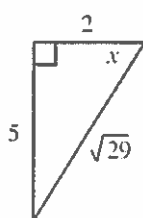
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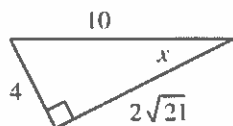
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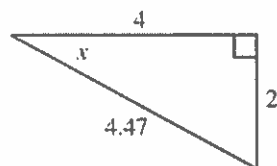
8.



9.



10.



We can find the sine of an angle even if only two sides of the triangle are known. If the two sides are the hypotenuse and opposite side, then we plug them into  $\sin(x) = \frac{O}{H}$  directly. Otherwise, we will need to use the Pythagorean theorem to find the length of the missing side.

**Example**

Find the sine of angle  $a$  from Figure 5.19.

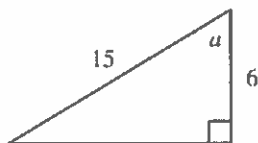


Figure 5.19

We know that the hypotenuse is  $H = 15$ , but we don't know the length  $O$  of the opposite side. By the Pythagorean theorem,

$$6^2 + O^2 = 15^2$$

$$O = \sqrt{189} = 3\sqrt{21}$$

$$\text{Thus, } \sin(a) = \frac{3\sqrt{21}}{15} = \frac{\sqrt{21}}{5}.$$

**Example**

Find the sine of angle  $x$  in Figure 5.20.

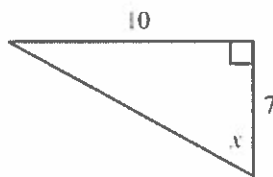


Figure 5.20

The opposite side has length  $O = 10$ . The hypotenuse  $H$  can be found by the Pythagorean theorem:

$$H^2 = 10^2 + 7^2$$

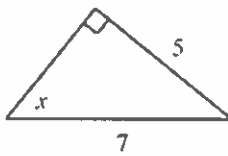
$$H = \sqrt{149}$$

This means that  $\sin(x) = \frac{10}{\sqrt{149}} = \frac{10\sqrt{149}}{149}$ .

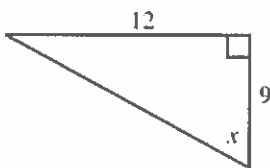
### ► Practice

Find the sine of the angle  $x$  in each triangle.

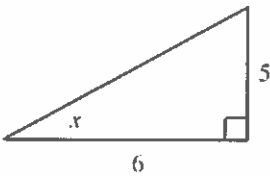
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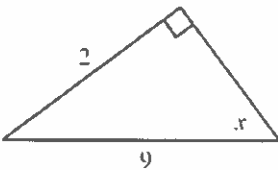
12.



13.

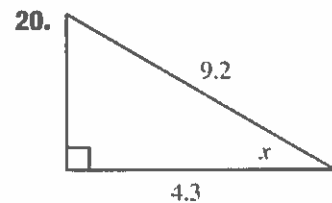
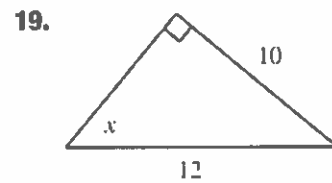
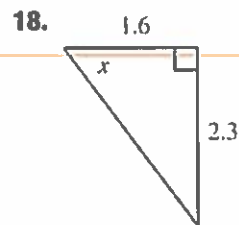
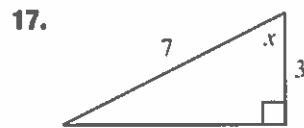
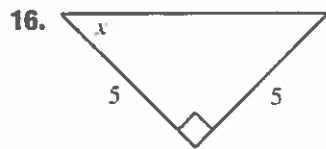


14.



15.





If we are given the sine of an angle  $x$ , we can't immediately name the sides of a right triangle with angle  $x$ . This is because the triangle could be large or small, as discussed at the beginning of this lesson. However, if we are given even one of the sides, then we can figure out the lengths of the other sides.

### Example

Suppose the angle  $x$  in Figure 5.31 has  $\sin(x) = \frac{3}{5}$ . What is the hypotenuse of this triangle?

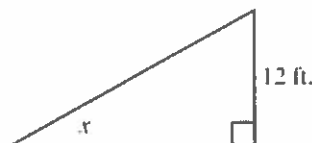


Figure 5.31



The side opposite the angle  $x$  has length 12. If the length of the hypotenuse is  $H$ , then  $\sin(x) = \frac{O}{H} = \frac{12}{H}$ . Because we know that  $\sin(x) = \frac{3}{5}$ , it follows that

$$\frac{12}{H} = \frac{3}{5}$$

If we cross multiply, we get

$$60 = 3H$$

$$H = 20$$

Thus, the hypotenuse of this triangle is 20 feet long.

### Example

Suppose the hypotenuse of a right triangle is 35 feet long. If the sine of one of the non-right angles is  $\sin(x) = 0.73$ , then what are the lengths of the other sides?

We can draw a right triangle with an angle of measure  $x$  and a hypotenuse of length 35. If we label the side opposite the angle  $x$  with variable  $O$ , and the side adjacent to (next to) angle  $x$  by  $A$ , then we get the picture in Figure 5.32.

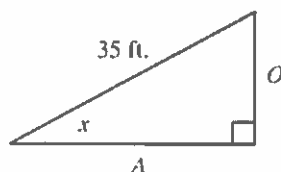


Figure 5.32

We know that  $\sin(x) = \frac{O}{35} = 0.73$ , so  $O = 35 \cdot (0.73) = 25.55$ .

To find the length of the adjacent side, we use the Pythagorean theorem:

$$A^2 + (25.55)^2 = 35^2$$

$$A = \sqrt{572.1975} \approx 23.92$$

Thus, the lengths of the other two sides are about 25.55 feet and 23.92 feet.

### ► Practice

21. If  $\sin(x) = \frac{2}{3}$  for the angle  $x$  in Figure 5.33, what is the length  $O$  of the opposite side?

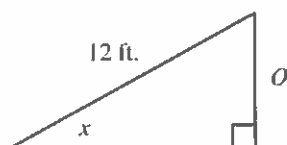


Figure 5.33

22. If  $\sin(x) = 0.8$ , what is the hypotenuse of the triangle in Figure 5.34?

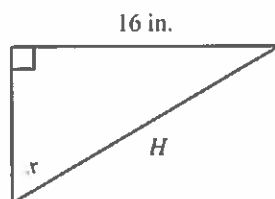


Figure 5.34

23. If  $\sin(x) = \frac{1}{6}$ , what is the length  $A$  of the side adjacent to angle  $x$  in Figure 5.35?

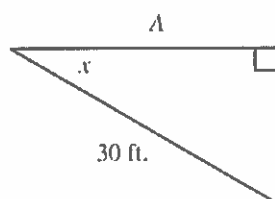


Figure 5.35

24. Suppose  $\sin(x) = 0.65$  for the angle in Figure 5.36. What is the length of the side adjacent to the angle  $x$ ?

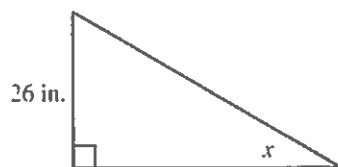


Figure 5.36

25. Suppose the hypotenuse of a triangle with angle  $x$  is  $H = 15$  inches long. If  $\sin(x) = \frac{2}{5}$ , what is the length of the side opposite  $x$ ?

## LESSON

# 6



# Cosine

In this lesson, we study a new trigonometric function, cosine. Just as with sine, this function is based on a right triangle. For cosine, however, we use a different pair of sides from that triangle. We find the cosine for the angles of various triangles, and then examine the relationships between sine and cosine.

**Cosine** is traditionally viewed as the second trigonometric function. The domain of this function consists of all angles  $x$ , just like the sine function. As with sine, we will use angles only between  $0^\circ$  and  $90^\circ$  until Lesson 11. This time, when we have a right triangle with angle  $x$ , we use the **adjacent** side  $A$  instead of the opposite side. The adjacent side is the leg of the triangle that forms the angle  $x$ , together with the hypotenuse, as shown in Figure 6.1.

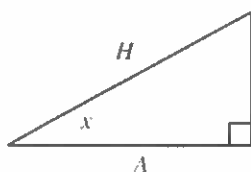


Figure 6.1

$$\cos(x) = \frac{A}{H}$$

### Example

Find the cosine of the angle  $x$  in Figure 6.2.

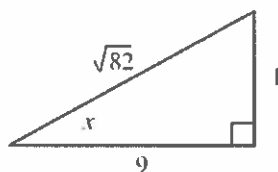


Figure 6.2

Here, the hypotenuse  $H = \sqrt{82}$  and the adjacent side  $A = 9$ , so

$$\cos(x) = \frac{9}{\sqrt{82}} = \frac{9\sqrt{82}}{82}$$

### Example

Find the cosine of the angle  $\theta$  in Figure 6.3.

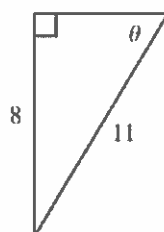


Figure 6.3

We need to use the Pythagorean theorem to find the length of the adjacent side  $A$ .

$$A^2 + 8^2 = 11^2$$

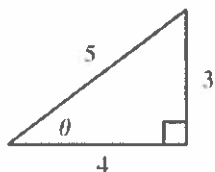
$$A = \sqrt{57}$$

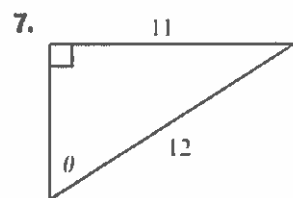
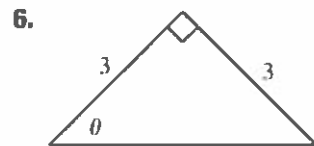
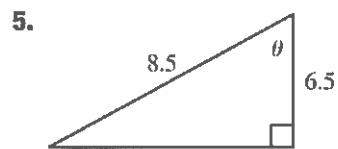
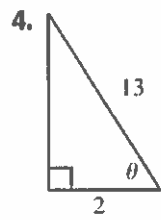
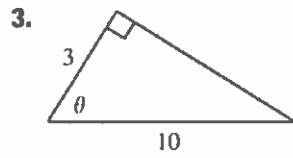
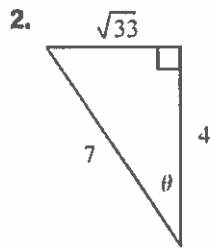
$$\cos(\theta) = \frac{\sqrt{57}}{11}$$

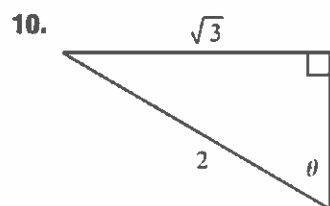
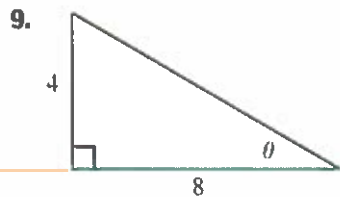
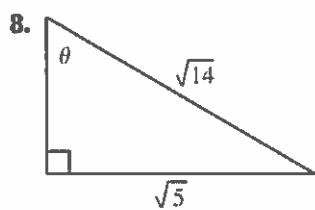
### ► Practice

Find the cosine of the angle  $\theta$  in each triangle.

1.







### ► The Relationship between Sine and Cosine

There are several relationships between the sine and cosine functions. Suppose that a right triangle with angle  $x$  has hypotenuse  $H$  and legs  $A$  and  $O$  as depicted in Figure 6.14.

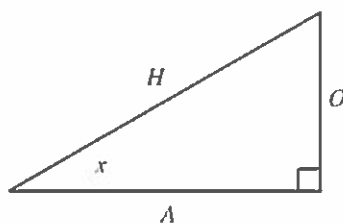


Figure 6.14

$$\sin(x) = \frac{O}{H} \text{ and } \cos(x) = \frac{A}{H}$$

The third, unmarked, angle of this triangle measures  $90^\circ - x$  (or  $\frac{\pi}{2} - x$  in radians) because the angles of a triangle must sum to  $180^\circ$  (or  $\pi$  radians). This is illustrated in Figure 6.15.

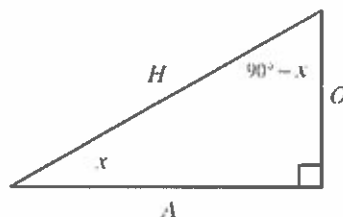


Figure 6.15

Here,  $\sin(90^\circ - x) = \frac{A}{H}$  because the side opposite the  $90^\circ - x$  angle is  $A$ , the one that is adjacent to angle  $x$ . This means that

$$\cos(x) = \sin(90^\circ - x)$$

This is the first of the relationships between sine and cosine.

When two angles add up to  $90^\circ$ , like  $x$  and  $90^\circ - x$ , they are called **complements**. Thus, the cosine of an angle  $x$  equals the sine of the complement  $90^\circ - x$ . In fact, the phrase *sine of the complement* is the origin of the word *cosine*.

$$\text{Similarly, } \cos(90^\circ - x) = \frac{O}{H} = \sin(x).$$

### Example

If  $\sin(20^\circ) \approx 0.342$ , then what is  $\cos(70^\circ)$ ?

Because  $\cos(x) = \sin(90^\circ - x)$ , it follows that  $\cos(70^\circ) = \sin(90^\circ - 70^\circ) = \sin(20^\circ) \approx 0.342$ .

The next relationship between sine and cosine comes from the Pythagorean theorem. Start with a right triangle with angle  $x$  and sides  $O$ ,  $A$ , and  $H$  as shown in Figure 6.16.

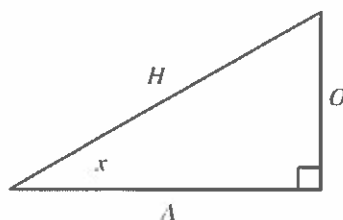


Figure 6.16

By the Pythagorean theorem,  $A^2 + O^2 = H^2$ . If we divide both sides of the equation by  $H^2$ , we get:

$$\frac{A^2 + O^2}{H^2} = \frac{H^2}{H^2}$$

$$\frac{A^2}{H^2} + \frac{O^2}{H^2} = 1$$

$$\left(\frac{A}{H}\right)^2 + \left(\frac{O}{H}\right)^2 = 1$$

$$(\sin(x))^2 + (\cos(x))^2 = 1$$

$$\sin^2(x) + \cos^2(x) = 1$$

This is the main relationship between  $\sin(x)$  and  $\cos(x)$ .

### Tip

When a trigonometric function like sine or cosine is raised to a power, the exponent is usually put right after the function's name. Thus,  $\sin^2(x) = (\sin(x))^2$  and  $\cos^5(x) = (\cos(x))^5$ .

### Example

If  $\sin(x) = \frac{2}{3}$ , then what is  $\cos(x)$ ?

We can solve this by using the formula  $\sin^2(x) + \cos^2(x) = 1$ .

$$\left(\frac{2}{3}\right)^2 + \cos^2(x) = 1$$

$$\cos^2(x) = 1 - \frac{4}{9} = \frac{5}{9}$$

$$\cos(x) = \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3}$$

### ► Practice

11. If  $\sin(x) = \frac{2}{5}$ , what is  $\cos(90^\circ - x)$ ?
12. If  $\sin(x) = \frac{2}{5}$ , what is  $\cos(x)$ ?
13. If  $\sin(30^\circ) = \frac{1}{2}$ , then what is  $\cos(60^\circ)$ ?
14. What is  $\cos(30^\circ)$  if  $\sin(30^\circ) = \frac{1}{2}$ ?



15. Find  $\cos(x)$  when  $\sin(x) = \frac{1}{10}$ .
16. If  $\cos(40^\circ) \approx 0.766$ , then what is  $\sin(50^\circ)$ ?
17. If  $\cos(x) = \frac{3}{4}$ , then what is  $\sin(x)$ ?
18. Find  $\sin(x)$  when  $\cos(x) = \frac{2}{7}$ .
19. If  $\cos(45^\circ) = \frac{\sqrt{2}}{2}$ , then what is  $\sin(45^\circ)$ ?
20. What is the  $\sin(x)$  if  $\cos(x) = \frac{\sqrt{2}}{4}$ ?

If we are given the cosine of an angle from a right triangle and any length of the triangle, we can find either of the other sides.

### Example

If  $\cos(x) = 0.53$  for angle  $x$  in a right triangle with hypotenuse 10 feet, what are the lengths of the other two sides?  
We draw the right triangle, as in Figure 6.17.

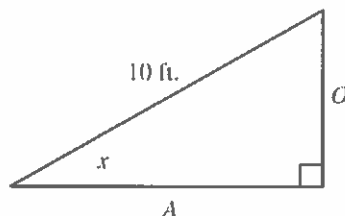


Figure 6.17

We know that  $\cos(x) = \frac{A}{10} = 0.53$  so the adjacent side is  $A = 5.3$  feet long. We use the Pythagorean theorem to find the third side  $O$ .

$$(5.3)^2 + O^2 = 10^2$$

$$O = \sqrt{71.91} \approx 8.48 \text{ feet long}$$

### Example

Suppose  $\cos(x) = \frac{3}{5}$  for angle  $x$  in a right triangle. If the length of the side opposite  $x$  is 12 inches long, what is the hypotenuse of the triangle?

Because  $\cos(x) = \frac{A}{H}$ , the length  $O = 12$  inches cannot be immediately used. The trick is to use  $\sin(x) = \frac{O}{H}$ . We must thus figure out what  $\sin(x)$  is when  $\cos(x) = \frac{3}{5}$ .

$$\sin^2(x) + \cos^2(x) = 1$$

$$\sin^2(x) + \left(\frac{3}{5}\right)^2 = 1$$

$$\sin(x) = \sqrt{1 - \frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

Thus,  $\sin(x) = \frac{4}{5} = \frac{12}{H}$ , so  $H = 15$ . The hypotenuse is 15 inches long.

### ► Practice

21. Suppose  $\cos(x) = 0.11$ . If the hypotenuse of a right triangle with angle  $x$  is  $H = 14$  feet long, what is the length of the side adjacent to angle  $x$ ?
22. Suppose  $\cos(x) = 0.5$  for angle  $x$  in a right triangle. If the side adjacent to angle  $x$  has length  $A = 3$  feet long, what is the length of the hypotenuse?
23. Suppose  $\cos(x) = \frac{2}{7}$  for angle  $x$  in a right triangle with hypotenuse  $H = 10$  inches long. What are the lengths of the other two sides?

## LESSON

# 7



# Tangent

In this lesson, we study the third trigonometric function, **tangent**. Tangent is much like sine and cosine, but it uses different sides of the right triangle. We evaluate the tangent for a number of angles, and examine the various relationships among sine, cosine, and tangent. We also see how knowing one of the trigonometric functions enables us to figure out the values of the others.

The process for evaluating the tangent of an angle is similar to that of sine and cosine:

Step one: Start with an angle of measure  $x$  (between  $0^\circ$  and  $90^\circ$ ).

Step two: Make a right triangle with angle  $x$ , as in Figure 7.1.

Step three: Measure the length  $O$  of the side opposite the angle  $x$  and  $A$ , the side adjacent.

Step four: The tangent of the angle  $x$  is the ratio of the two sides:

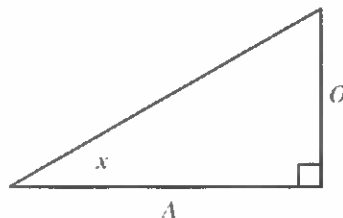


Figure 7.1

$$\tan(x) = \frac{O}{A}$$

As with the sine and cosine, it is really difficult for human beings to draw triangles with great precision, so it is very hard to calculate the tangent by actually constructing a triangle and measuring its sides. Instead, we exercise our understanding of tangent by calculating it for angles in already known triangles.

### Example

What is the tangent of the angle  $x$  in Figure 7.2?

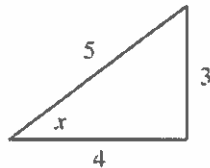


Figure 7.2

The tangent of an angle can only be found from the sides of a right triangle. This triangle happens to be right (even though it is not marked as so), because  $3^2 + 4^2 = 5^2$ . Thus,

$$O = 3, A = 4, \text{ and } \tan(x) = \frac{3}{4} = 0.75.$$

There is an interesting difference between the tangent trigonometric function and sine and cosine. Because the sine and cosine are ratios with  $H$ , the largest side, on the bottom, they could never be bigger than 1. The tangent, which compares the lengths of the two smaller sides, could be quite large.

### Example

What is the tangent of the angle  $x$  in Figure 7.3?



Figure 7.3

$$O = 4 \text{ and } A = 1, \text{ so } \tan(x) = \frac{4}{1} = 4.$$

### Example

Find the tangent of angle  $x$  in Figure 7.4.

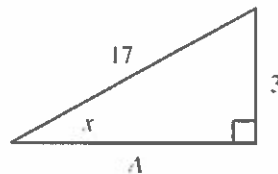


Figure 7.4

To find the length of the adjacent side  $A$ , we need to use the Pythagorean theorem:

$$A^2 + 3^2 = 17^2$$

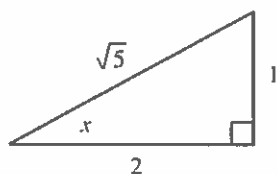
$$A = \sqrt{280} = 2\sqrt{70}$$

$$\text{Thus, } \tan(x) = \frac{O}{A} = \frac{3}{2\sqrt{70}} = \frac{3\sqrt{70}}{140}.$$

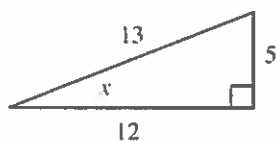
### ► Practice

Find the tangent of angle  $x$  in the following triangles.

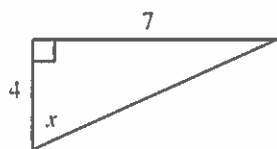
1.



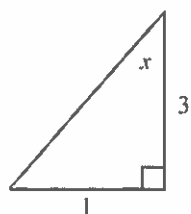
2.



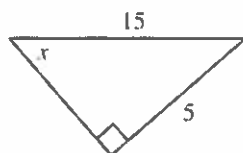
3.



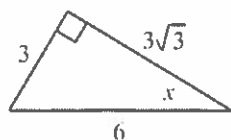
4.

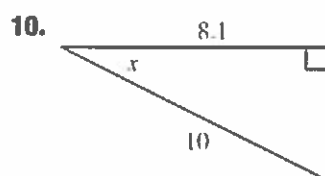
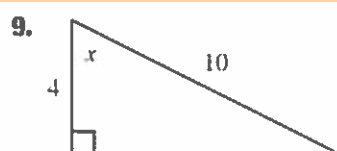
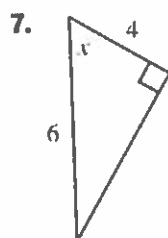


5.



6.





## ► The Relationship between Sine, Cosine, and Tangent

The main relationship between tangent, sine, and cosine of an angle  $x$  is

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

which can be seen easily from any right triangle with angle  $x$ , as shown in Figure 7.15.

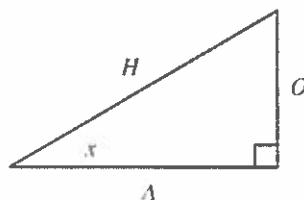


Figure 7.15

$$\frac{\sin(x)}{\cos(x)} = \frac{\frac{O}{H}}{\frac{A}{H}} = \frac{O}{H} \div \frac{A}{H} = \frac{O}{H} \times \frac{H}{A} = \frac{O}{A} = \tan(x)$$

### Tip

Remember that dividing by a fraction is the same as multiplying by its flip (**reciprocal**). For example,

$$10 \div \frac{3}{5} = 10 \times \frac{5}{3}$$

because they both become ten when you multiply by  $\frac{3}{5}$ .

$$\left(10 \div \frac{3}{5}\right) \times \frac{3}{5} = 10$$

because multiplication and division undo each other.

$$10 \times \frac{5}{3} \times \frac{3}{5} = \frac{150}{15} = 10$$

This is why  $\frac{O}{H} \div \frac{A}{H} = \frac{O}{H} \times \frac{H}{A} = \frac{O}{A}$ .

### Example

What is  $\tan(x)$  if  $\sin(x) = \frac{1}{3}$  and  $\cos(x) = \frac{\sqrt{8}}{3}$ ?

$$\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{1}{3} \div \frac{\sqrt{8}}{3} = \frac{1}{3} \times \frac{3}{\sqrt{8}} = \frac{1}{\sqrt{8}} = \frac{\sqrt{8}}{8}$$

### Example

If  $\tan(x) = \frac{2}{5}$  and  $\cos(x) = \frac{5\sqrt{29}}{29}$ , then what is  $\sin(x)$ ?

$$\tan(x) = \frac{\sin(x)}{\cos(x)}, \text{ so } \sin(x) = \cos(x) \cdot \tan(x) = \frac{2}{5} \cdot \frac{5\sqrt{29}}{29} = \frac{2\sqrt{29}}{29}$$

### ► Practice

11. What is  $\tan(x)$  if  $\sin(x) = \frac{3}{5}$  and  $\cos(x) = \frac{4}{5}$ ?
12. Suppose  $\sin(\theta) = \frac{\sqrt{5}}{5}$  and  $\cos(\theta) = \frac{2\sqrt{5}}{5}$ . What is  $\tan(\theta)$ ?
13. What is  $\tan(x)$  when  $\sin(x) = \frac{8}{17}$  and  $\cos(x) = \frac{15}{17}$ ?
14. If  $\cos(x) = \frac{\sqrt{17}}{17}$  and  $\sin(x) = \frac{4\sqrt{17}}{17}$ , then what is  $\tan(x)$ ?

15. If  $\sin(x) = 0.75$  and  $\cos(x) \approx 0.66$ , then what is  $\tan(x)$ , approximately?
16. If  $\sin(x) = \frac{\sqrt{2}}{2}$  and  $\cos(x) = \frac{\sqrt{2}}{2}$ , then what is  $\tan(x)$ ?
17. What is  $\sin(x)$  when  $\tan(x) = \frac{2}{3}$  and  $\cos(x) = \frac{3\sqrt{13}}{13}$ ?
18. What is  $\sin(x)$  when  $\cos(x) = \frac{\sqrt{91}}{10}$  and  $\tan(x) = \frac{3\sqrt{91}}{91}$ ?
19. What is  $\cos(x)$  when  $\sin(x) = \frac{4}{7}$  and  $\tan(x) = \frac{4\sqrt{33}}{33}$ ?
20. What is the cosine of angle  $\theta$  for which  $\tan(\theta) = \frac{\sqrt{21}}{2}$  and  $\sin(\theta) = \frac{\sqrt{21}}{5}$ ?

### ► SOH-CAH-TOA

Many students remember the foundations of trigonometry with one word: *SOH-CAH-TOA*, and the triangle in Figure 7.16.

$$\text{SOH stands for } \sin(x) = \frac{O}{H}$$

$$\text{CAH stands for } \cos(x) = \frac{A}{H}$$

$$\text{TOA stands for } \tan(x) = \frac{O}{A}$$

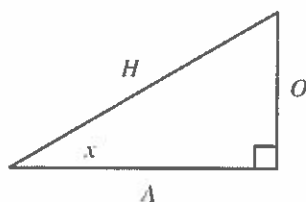


Figure 7.16

These three ratios are all interesting properties of the angle  $x$ . However, if you know any one of them, then you can figure out the other two.

### Example

Suppose the sine of an angle  $x$  is  $\sin(x) = 0.72$ . What would the cosine and tangent of this angle be?

All we need to do is draw a right triangle that has side lengths that make  $\sin(x) = 0.72 = \frac{72}{100}$ , like the one in Figure 7.17 (there is no need to draw to scale). We then find the other side by the Pythagorean theorem.



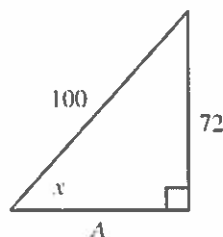


Figure 7.17

The adjacent side  $A$  is  $A = \sqrt{100^2 - 72^2} = \sqrt{4,816} = 4\sqrt{301}$ .

Thus, the cosine of the angle is  $\cos(x) = \frac{4\sqrt{301}}{100} = \frac{\sqrt{301}}{25}$  and the tangent is  $\tan(x) = \frac{72}{4\sqrt{301}} = \frac{18\sqrt{301}}{301}$ .

### Example

What is the sine and cosine of angle  $x$  if  $\tan(x) = \frac{2\sqrt{5}}{5}$ ?

The easiest way to make a right triangle with angle  $x$  so that  $\tan(x) = \frac{2\sqrt{5}}{5}$  is by making the opposite side  $O = 2\sqrt{5}$  and the adjacent side  $A = 5$ , as in Figure 7.18. Any choice of opposite and adjacent side length with the ratio  $\frac{2\sqrt{5}}{5}$  would work; because it is easier not to work with fractions, we have chosen the numerator to be the opposite side and the denominator to be the adjacent side.

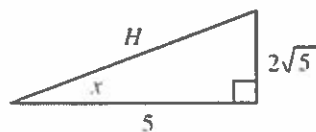


Figure 7.18

The hypotenuse  $H$  of this triangle is found by the Pythagorean theorem.

$$H^2 = 5^2 + (2\sqrt{5})^2$$

$$H = \sqrt{25 + 20} = \sqrt{45} = 3\sqrt{5}$$

$$\text{Thus, } \sin(x) = \frac{2\sqrt{5}}{3\sqrt{5}} = \frac{2}{3} \text{ and } \cos(x) = \frac{5}{3\sqrt{5}} = \frac{5\sqrt{5}}{15} = \frac{\sqrt{5}}{3}.$$

## ► Practice

21. What is  $\tan(x)$  when  $\sin(x) = \frac{45}{53}$ ?
22. If  $\cos(x) = \frac{2}{7}$ , then what is  $\tan(x)$ ?
23. If  $\cos(x) = \frac{5}{8}$ , then what is  $\sin(x)$ ?
24. Suppose  $\tan(x) = \frac{\sqrt{51}}{7}$ . What is  $\cos(x)$ ?
25. Suppose  $\sin(x) = \frac{\sqrt{11}}{5}$ . What is  $\cos(x)$ ?
26. If  $\sin(x) = \frac{12\sqrt{193}}{193}$ , then what is  $\tan(x)$ ?
27. What is the cosine of the angle whose tangent is  $\sqrt{\frac{8}{41}}$ ?

# LESSON

# 8



# Secant, Cosecant, and Cotangent

In this lesson, we introduce the last three trigonometric functions: **secant**, **cosecant**, and **cotangent**. Again, these are evaluated by dividing one length of a right triangle by another. We then examine the relationships among all six trigonometric functions. If we know the value of one trigonometric function, we will be able to figure out the values of the other five.

Suppose  $x$  is the angle of the right triangle in Figure 8.1. The length of the hypotenuse is  $H$ , the side opposite  $x$  is  $O$ , and the adjacent side has length  $A$ .

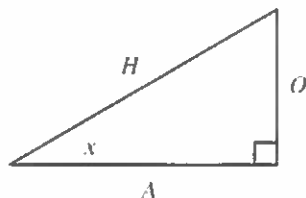


Figure 8.1

There are only six possible ratios that can be formed by a pair of these sides:

$$\frac{O}{H}, \frac{O}{A}, \frac{A}{H}, \frac{A}{O}, \frac{H}{A}, \text{ and } \frac{H}{O}$$

The first three of these have been covered already:

$$\sin(x) = \frac{O}{H}, \tan(x) = \frac{O}{A}, \text{ and } \cos(x) = \frac{A}{H}$$

Just for the sake of completeness, we define three more trigonometric functions for the last three: cotangent, secant, and cosecant.

$$\cot(x) = \frac{A}{O}$$

$$\sec(x) = \frac{H}{A}$$

$$\csc(x) = \frac{H}{O}$$

### Example

What is the secant, cosecant, and cotangent of the angle  $x$  in Figure 8.2?

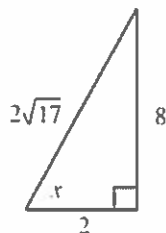


Figure 8.2

Here,  $H = 2\sqrt{17}$ ,  $O = 8$ , and  $A = 2$ , so

$$\sec(x) = \frac{H}{A} = \frac{2\sqrt{17}}{2} = \sqrt{17}$$

$$\csc(x) = \frac{H}{O} = \frac{2\sqrt{17}}{8} = \frac{\sqrt{17}}{4}$$

$$\cot(x) = \frac{A}{O} = \frac{2}{8} = \frac{1}{4}$$

## ► Practice

Evaluate the following trigonometric functions for the angles  $w$ ,  $x$ ,  $y$ , and  $z$  in Figure 8.3.

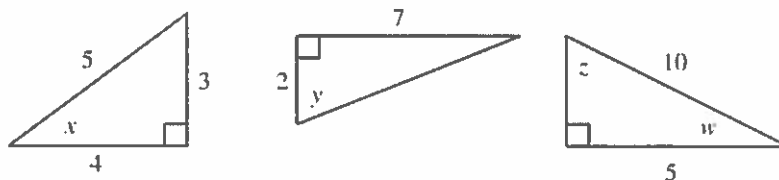


Figure 8.3

1.  $\sec(x)$
2.  $\csc(x)$
3.  $\cot(x)$
4.  $\cot(y)$
5.  $\sec(y)$
6.  $\csc(y)$
7.  $\cot(z)$
8.  $\tan(w)$
9.  $\sec(z)$
10.  $\csc(w)$

## ► Formulas for Secant, Cosecant, and Cotangent

In Lesson 6, we saw that the cosine of an angle is the sine of the complementary angle. This can be written as:

$$\cos(x) = \sin(90^\circ - x)$$

The same is true for the other cofunctions: cotangent and cosecant. To see this, look at the angle  $x$  and its complement  $90^\circ - x$  in Figure 8.4.

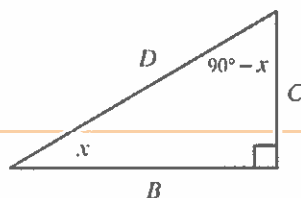


Figure 8.4

For angle  $x$ , the opposite side has length  $C$  and the adjacent side has length  $B$ . Thus,

$$\cot(x) = \frac{B}{C} \text{ and } \csc(x) = \frac{D}{C}$$

For angle  $90^\circ - x$ , the opposite side has length  $B$  and the adjacent side has length  $C$ . Thus,

$$\tan(90^\circ - x) = \frac{B}{C} \text{ and } \sec(90^\circ - x) = \frac{D}{C}$$

It follows that

$$\cot(x) = \tan(90^\circ - x)$$

and

$$\csc(x) = \sec(90^\circ - x)$$

These properties explain the origins of the names for cosecant and cotangent. *Cosecant* means “the secant of the complementary angle,” and *cotangent* means “the tangent of the complementary angle.”

Similarly,  $\tan(x) = \cot(90^\circ - x)$  and  $\sec(x) = \csc(90^\circ - x)$ . (Remember that  $90^\circ = \frac{\pi}{2}$  radians, so each  $90^\circ - x$  should be replaced with  $\frac{\pi}{2} - x$  if the angle  $x$  is measured in radians.)

While all of these formulas are quite lovely, there are others that are more basic and useful. Take the angle  $x$  from the right triangle in Figure 8.5.

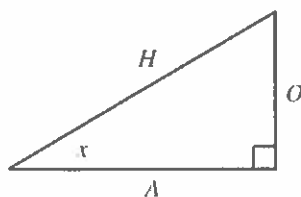


Figure 8.5

## SECANT, COSECANT, AND COTANGENT

The reciprocal (flip) of  $\cos(x)$  is

$$\frac{1}{\cos(x)} = \frac{1}{\frac{A}{H}} = 1 \div \frac{A}{H} = 1 \cdot \frac{H}{A} = \frac{H}{A} = \sec(x)$$

Similarly,

$$\frac{1}{\sin(x)} = \frac{1}{\frac{O}{H}} = \frac{H}{O} = \csc(x)$$

Finally,

$$\frac{1}{\tan(x)} = \frac{1}{\frac{O}{A}} = \frac{A}{O} = \cot(x)$$

### Tip

This is how most people remember the last three trigonometric functions:

$$\sec(x) = \frac{1}{\cos(x)}, \csc(x) = \frac{1}{\sin(x)}, \text{ and } \cot(x) = \frac{1}{\tan(x)}$$

Remember from Lesson 7 that  $\tan(x) = \frac{\sin(x)}{\cos(x)}$ . This means that cotangent is also

$$\cot(x) = \frac{1}{\tan(x)} = \frac{1}{\frac{\sin(x)}{\cos(x)}} = \frac{\cos(x)}{\sin(x)}$$

Thus, every trigonometric function can be written entirely in terms of  $\sin(x)$  and  $\cos(x)$ .

### Example

How can  $\csc(x) \cdot \tan(x)$  be represented entirely in terms of  $\sin(x)$  and  $\cos(x)$ ?

$$\csc(x) \cdot \tan(x) = \frac{1}{\sin(x)} \cdot \frac{\sin(x)}{\cos(x)} = \frac{1}{\cos(x)}$$

This is, of course, just  $\sec(x)$ , but sometimes  $\frac{1}{\cos(x)}$  is easier to understand.

### Example

What is  $\frac{\sec(x)}{\cos(x)}$  in terms of  $\sin(x)$  and  $\cos(x)$ ?

$$\frac{\sec(x)}{\cos(x)} = \frac{1}{\cos(x)} = \frac{1}{\cos(x)} \div \cos(x) = \frac{1}{\cos(x)} \cdot \frac{1}{\cos(x)} = \frac{1}{\cos^2(x)}$$

### Example

Represent  $\cot(90^\circ - x) \cdot \sec(x)$  in terms of  $\sin(x)$  and  $\cos(x)$ . Here we first use  $\cot(90^\circ - x) = \tan(x)$ , so

$$\cot(90^\circ - x) \cdot \sec(x) = \tan(x) \cdot \sec(x) = \frac{\sin(x)}{\cos(x)} \cdot \frac{1}{\cos(x)} = \frac{\sin(x)}{\cos^2(x)}$$

## ► Practice

Represent each of the following in terms of  $\sin(x)$  and  $\cos(x)$ . You may assume that each angle  $x$  is measured in degrees.

11.  $\sec(x) \cdot \tan(x)$

12.  $\cot(x) \cdot \sin(x)$

13.  $\cos^2(x) \cdot \tan(x)$

14.  $\frac{\csc(x)}{\cot(x)}$

15.  $\frac{\sec(x) \cdot \csc(x)}{\sin^2(x)}$

16.  $\sec(90^\circ - x)$

17.  $\cos(90^\circ - x) \cdot \sec(x)$

18.  $\frac{\tan(x)}{\tan(90^\circ - x)}$

19.  $\cot(x) \cdot \csc(90^\circ - x) \cdot \sin(x)$

20.  $\sin(90^\circ - x) \cdot \cot(90^\circ - x)$



### ► More Formulas

In Lesson 6, we used the Pythagorean theorem to prove that

$$\sin^2(x) + \cos^2(x) = 1$$

If we divide both sides of this equation by  $\cos^2(x)$ , we get

$$\begin{aligned}\frac{\sin^2(x) + \cos^2(x)}{\cos^2(x)} &= \frac{1}{\cos^2(x)} \\ \frac{\sin^2(x)}{\cos^2(x)} + \frac{\cos^2(x)}{\cos^2(x)} &= \left(\frac{1}{\cos(x)}\right)^2 \\ \left(\frac{\sin(x)}{\cos(x)}\right)^2 + 1 &= \sec^2(x) \\ \tan^2(x) + 1 &= \sec^2(x)\end{aligned}$$

Similarly, if we divide both sides of  $\sin^2(x) + \cos^2(x) = 1$  by  $\sin^2(x)$ , we get

$$\begin{aligned}\frac{\sin^2(x)}{\sin^2(x)} + \frac{\cos^2(x)}{\sin^2(x)} &= \frac{1}{\sin^2(x)} \\ 1 + \cot^2(x) &= \csc^2(x)\end{aligned}$$

With these formulas, we can find one trigonometric value of an angle  $x$  given another one.

#### Example

If  $\cot(x) = \frac{5}{2}$ , then what is  $\csc(x)$ ?

$$\begin{aligned}1 + \cot^2(x) &= \csc^2(x) \\ 1 + \left(\frac{5}{2}\right)^2 &= \csc^2(x) \\ \csc(x) &= \sqrt{1 + \frac{25}{4}} = \sqrt{\frac{29}{4}} = \frac{\sqrt{29}}{2}\end{aligned}$$

A more general technique for solving such problems was introduced in Lesson 7:

- Draw a right triangle with angle  $x$ .
- Use the known trigonometric value to assign lengths to two sides of the triangle.
- Use the Pythagorean theorem to find the third side.
- Use the triangle to find all the other trigonometric values.

**Example**

If  $\cot(x) = \frac{5}{2}$ , then what are the other five trigonometric values of  $x$ ?

Draw a right triangle with angle  $x$ . If the side adjacent to angle  $x$  is  $A = 5$  and the side opposite is  $O = 2$ , then this will ensure that  $\cot(x) = \frac{A}{O} = \frac{5}{2}$ . See Figure 8.6.

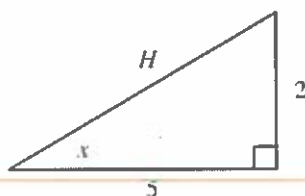


Figure 8.6

The hypotenuse  $H$  is found with the Pythagorean theorem:

$$2^2 + 5^2 = H^2$$

$$H = \sqrt{29}$$

Thus,  $\sin(x) = \frac{2}{\sqrt{29}} = \frac{2\sqrt{29}}{29}$ ,  $\cos(x) = \frac{5}{\sqrt{29}} = \frac{5\sqrt{29}}{29}$ ,  $\tan(x) = \frac{2}{5}$ ,  $\sec(x) = \frac{\sqrt{29}}{5}$ , and  $\csc(x) = \frac{\sqrt{29}}{2}$ .

**Example**

If  $\cos(x) = \frac{\sqrt{7}}{5}$ , then what is  $\csc(x)$ ?

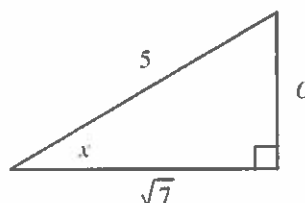


Figure 8.7

The angle  $x$  in Figure 8.7 has  $\cos(x) = \frac{\sqrt{7}}{5}$ . The side  $O$  opposite  $x$  is found with the Pythagorean theorem:

$$(\sqrt{7})^2 + O^2 = 5^2$$

$$O = \sqrt{18} = 3\sqrt{2}$$

Thus,  $\csc(x) = \frac{5}{3\sqrt{2}} = \frac{5\sqrt{2}}{6}$ .

**► Practice**

**21.** If  $\sec(x) = \frac{4}{3}$ , then what is  $\sin(x)$ ?

**22.** What is  $\csc(x)$  if  $\cot(x) = 4$ ?

**23.** Find  $\sec(x)$  for the angle  $x$  with  $\sin(x) = \frac{1}{2}$ .

**24.** If  $\cos(x) = \frac{\sqrt{3}}{3}$ , then what is  $\cot(x)$ ?

**25.** What is the sine of the angle with a cosecant of  $\frac{6}{5}$ ?

**26.** If  $\tan(x) = \frac{8}{3}$ , what is  $\sec(x)$ ?

**27.** What is the cotangent of angle  $x$  if  $\sec(x) = \frac{\sqrt{7}}{2}$ ?



LESSON

9



# Nice Triangles

In this lesson, we study two special right triangles for which we are able to know both the angle measures and the trigonometric values. The first triangle is the isosceles right triangle, with two  $45^\circ$  angles. The other triangle has angles of measure  $30^\circ$ ,  $60^\circ$ , and  $90^\circ$ , and is half of an equilateral triangle.

So far, we have used right triangles to find  $\sin(x)$ ,  $\cos(x)$ , and other trigonometric values without knowing the measure (degrees or radians) of the angle  $x$ . There are only a few **nice triangles** for which the measures of the angles and the measures of the sides can both be known exactly.

## ► The 45-45 Right Triangle

The first of these is the isosceles right triangle. Because the hypotenuse is the longest side of a right triangle, it must be the legs that have the same length. Suppose the legs of this triangle each have a length of 1, as illustrated in Figure 9.1.

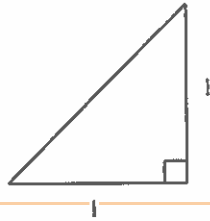


Figure 9.1

When a triangle has two sides of the same length, it will also have two angles of the same measure, as shown in Figure 9.2.

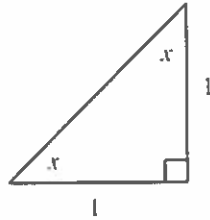


Figure 9.2

Because the angles of a triangle must add up to  $180^\circ$ , we calculate

$$x + x + 90^\circ = 180^\circ$$

$$x = 45^\circ$$

$$x = 45^\circ \cdot \frac{\pi}{180^\circ} = \frac{\pi}{4} \text{ radians}$$

Thus, the angle of an isosceles right triangle measures  $45^\circ$  or  $\frac{\pi}{4}$  radians.

The hypotenuse of this triangle can be calculated by the Pythagorean theorem:

$$1^2 + 1^2 = H^2$$

$$H = \sqrt{2}$$

With this, as shown in Figure 9.3, we can calculate all the trigonometric values for  $x = 45^\circ$  and  $x = \frac{\pi}{4}$ .

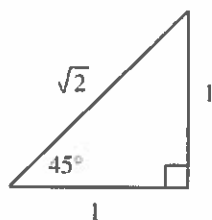


Figure 9.3

### Example

Find  $\sin(45^\circ)$ .

$$\sin(45^\circ) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

### Example

Find  $\sec\left(\frac{\pi}{4}\right)$ .

$$\sec\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{1} = \sqrt{2}$$

## ► Practice

Evaluate the following trigonometric functions.

1.  $\cos(45^\circ)$

2.  $\tan(45^\circ)$

3.  $\cot\left(\frac{\pi}{4}\right)$

4.  $\csc(45^\circ)$

5.  $\sin\left(\frac{\pi}{4}\right)$

6.  $\cos\left(\frac{\pi}{4}\right)$

### ► The 30-60 Right Triangle

The other nice triangle is half of an equilateral triangle. If each side of an equilateral triangle has a length of 1, we have the triangle in Figure 9.4. Recall that all angles of an equilateral triangle are  $60^\circ$  or  $\frac{\pi}{3}$  radians.

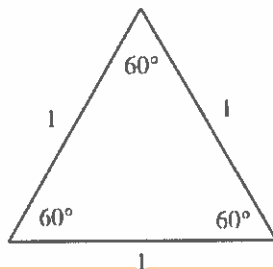


Figure 9.4

This is not a right triangle, so it cannot be used to calculate trigonometric values. However, if we split the triangle directly in half, we will obtain a right triangle, as depicted in Figure 9.5.

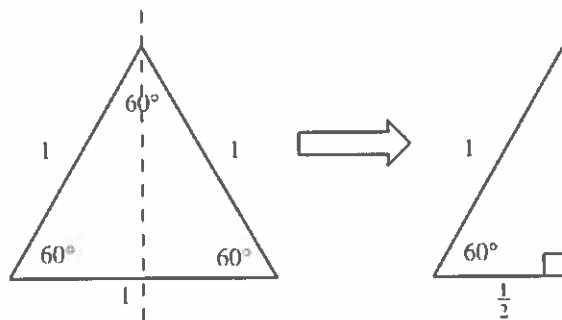


Figure 9.5

With the Pythagorean theorem, we can calculate the length  $O$  of the side opposite the  $60^\circ$  angle.

$$\left(\frac{1}{2}\right)^2 + O^2 = 1^2$$

$$\frac{1}{4} + O^2 = 1$$

$$O = \sqrt{1 - \frac{1}{4}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

We can also calculate the measure of the third angle  $y$ :

$$60^\circ + y + 90^\circ = 180^\circ$$

$$y = 30^\circ$$

$$y = 30^\circ \cdot \frac{\pi}{180^\circ} = \frac{\pi}{6}$$



With this, depicted in Figure 9.6, we can find all the trigonometric values of  $x = 60^\circ$ ,  $x = \frac{\pi}{3}$ ,  $x = 30^\circ$ , and  $x = \frac{\pi}{6}$ .

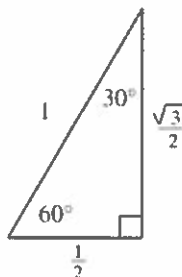


Figure 9.6

**Example**

Evaluate  $\sin(60^\circ)$ .

$$\sin(60^\circ) = \frac{\sqrt{3}}{2}$$

**Example**

Find  $\cot(30^\circ)$ .

$$\cot(30^\circ) = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \frac{\sqrt{3}}{2} \cdot \frac{2}{1} = \sqrt{3}$$

**Example**

What is  $\sec\left(\frac{\pi}{3}\right)$ ? First remember that  $\frac{\pi}{3}$  is the same as  $60^\circ$ .

$$\sec\left(\frac{\pi}{3}\right) = \sec(60^\circ) = 2$$

**► Practice**

Evaluate each of the following trigonometric functions.

7.  $\cos(60^\circ)$

8.  $\tan(60^\circ)$

9.  $\sin(30^\circ)$

10.  $\cot(60^\circ)$

11.  $\csc\left(\frac{\pi}{3}\right)$

12.  $\cos\left(\frac{\pi}{3}\right)$

13.  $\sec\left(\frac{\pi}{6}\right)$

14.  $\cos(30^\circ)$

15.  $\csc(60^\circ)$

16.  $\sin\left(\frac{\pi}{6}\right)$

17.  $\tan\left(\frac{\pi}{3}\right)$

18.  $\cot\left(\frac{\pi}{3}\right)$

19.  $\cos\left(\frac{\pi}{6}\right)$

20.  $\csc\left(\frac{\pi}{6}\right)$

21.  $\tan\left(\frac{\pi}{6}\right)$

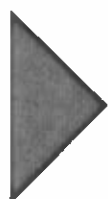
22.  $\sin\left(\frac{\pi}{3}\right)$

23.  $\cot\left(\frac{\pi}{6}\right)$

24.  $\sec(30^\circ)$

# LESSON

# 10



# The Unit Circle

In this lesson, we find the coordinates of many of the points on the unit circle. We use the nice triangles discussed in Lesson 9. In Lesson 11, these coordinates are used to find the trigonometric values for angles above  $90^\circ$  and below  $0^\circ$ .

So far, we have evaluated trigonometric functions only for acute angles, those between  $0^\circ$  and  $90^\circ$ . To work with other angles, we use the unit circle. Recall that a **unit circle** is a circle whose radius is one unit. A unit circle is graphed on the coordinate plane in Figure 10.1.

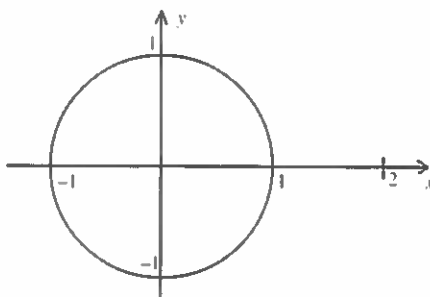


Figure 10.1

Every point  $(x, y)$  on the unit circle makes an angle with the origin  $(0, 0)$  and the positive  $x$ -axis. This angle  $\theta$  is measured counterclockwise, as illustrated in Figure 10.2.

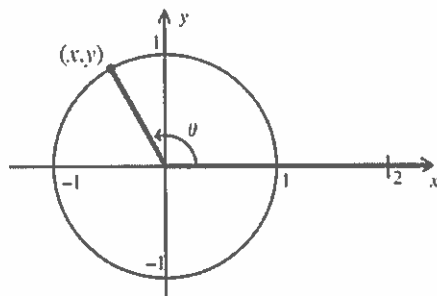


Figure 10.2

**Example**

What angle does the point  $(0, 1)$  make?

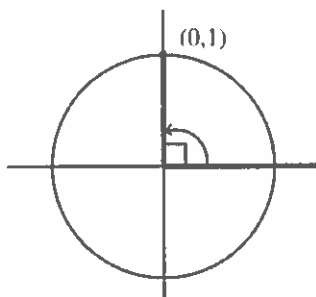


Figure 10.3

The point  $(0, 1)$  makes a right angle, as shown in Figure 10.3, either  $90^\circ$ , or  $\frac{\pi}{2}$ , radians.

**Example**

What angle is defined by the point  $(0, -1)$ ?

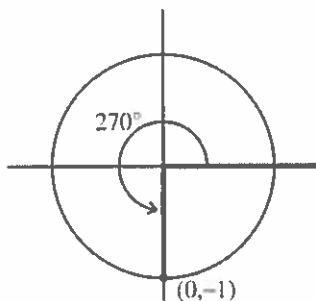


Figure 10.4

The point  $(0, -1)$  makes a  $270^\circ = \frac{3\pi}{2}$  radian angle, as shown in Figure 10.4.

The point  $(1,0)$  could be associated with either  $0^\circ$  or  $360^\circ$ . In fact, each point could be associated with many different angles if we allow angles greater than  $360^\circ$  (which go all the way around the circle) and negative angles (which go around clockwise, against the usual direction).

### Example

What point corresponds to an angle of  $\frac{5\pi}{2}$  radians?

$$\frac{5\pi}{2} \text{ radians is equivalent to } \frac{5\pi}{2} \cdot \frac{180^\circ}{\pi} = 450^\circ$$

This is a full circle  $360^\circ$  plus a quarter-turn  $90^\circ$  more, as shown in Figure 10.5.

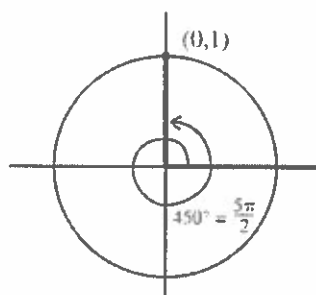


Figure 10.5

Thus, the angle  $\frac{5\pi}{2}$  corresponds to the point  $(0,1)$ .

Whenever an angle more than  $360^\circ$  is given, you can subtract  $360^\circ$  (or  $2\pi$ ) from it to remove a full turn. The same point of the unit circle will correspond. In the last example, for instance, we could have used  $450^\circ - 360^\circ = 90^\circ$  and gotten the same point  $(0,1)$ .

### Example

What point corresponds to the angle  $-\pi$ ?

$-\pi$  is equivalent to  $-180^\circ$ . This corresponds to the point  $(-1,0)$ , as shown in Figure 10.6. Notice that we go clockwise because the angle is negative.

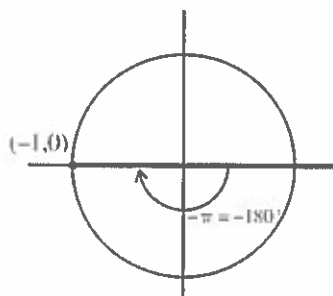


Figure 10.6

## ► Practice

1. Give the angle between  $0^\circ$  and  $360^\circ$  that corresponds to the point  $(-1,0)$ .
2. What negative angle between  $0^\circ$  and  $-360^\circ$  corresponds to the point  $(0,-1)$ ?

For problems 3 through 8, find the point on the unit circle that corresponds to the given angle.

3.  $\frac{\pi}{2}$

4.  $540^\circ$

5.  $-\frac{3\pi}{2}$

6.  $4\pi$

7.  $630^\circ$

8.  $-90^\circ$

► Multiples of  $60^\circ$ 

All of the angles thus far have been multiples of  $90^\circ$ . We can find the point that corresponds to  $60^\circ$  by using the nice 30-60-90 triangle from Lesson 9, shown in Figure 10.7.

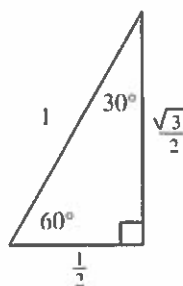


Figure 10.7

Because this has a hypotenuse of 1, it fits exactly into the unit circle of radius 1, as depicted in Figure 10.8.

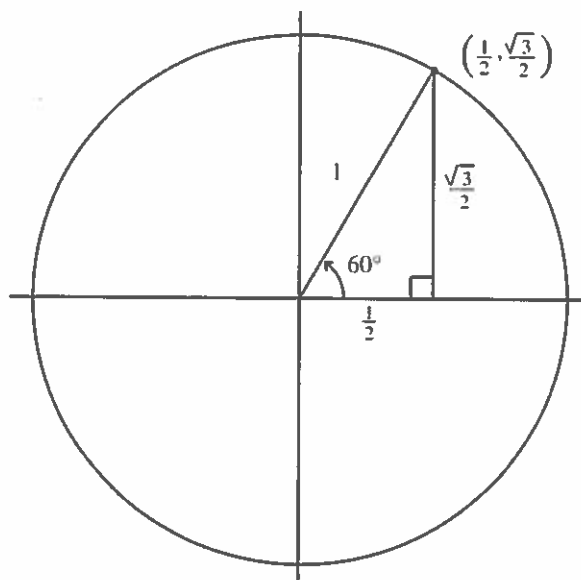


Figure 10.8

The point that corresponds to  $60^\circ$  is  $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$  because it is  $\frac{1}{2}$  a unit to the right of the origin and  $\frac{\sqrt{3}}{2}$  units up.

If we flip this triangle across the  $y$ -axis, we can find the point that corresponds to  $120^\circ$ , as shown in Figure 10.9.

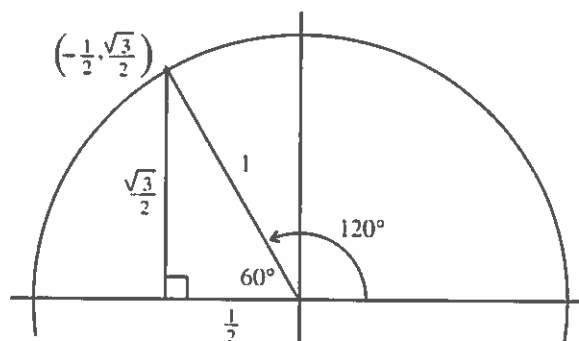


Figure 10.9

The  $120^\circ$  angle corresponds with point  $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ . Notice that the  $x$ -coordinate is negative because the point is to the left of the origin.

By flipping the triangle below the  $x$ -axis, we can see that  $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$  corresponds to  $240^\circ$  and  $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$  corresponds to  $300^\circ$ , as shown in Figure 10.10.

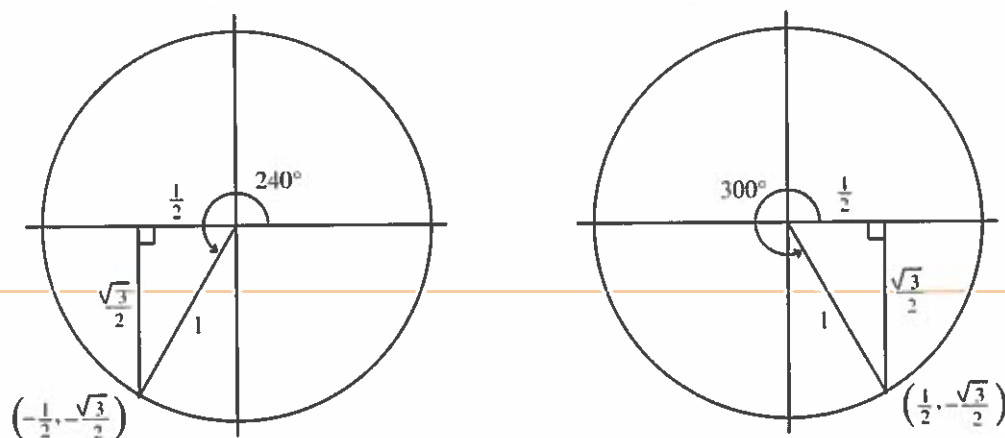


Figure 10.10

**Example**

What point on the unit circle corresponds to  $-\frac{4\pi}{3}$ ?

$$-\frac{4\pi}{3} \text{ is equivalent to } -\frac{4\pi}{3} \cdot \frac{180^\circ}{\pi} = -240^\circ$$

If we add a full rotation of  $360^\circ$  to this angle, it will still point in the same direction. Thus, the corresponding point for this angle is the same as that for  $-240^\circ + 360^\circ = 120^\circ$ . Thus, we can find the point either by going clockwise  $240^\circ$  or counterclockwise  $120^\circ$ , as shown in Figure 10.11. In either case, this angle corresponds to the point  $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$  on the unit circle.

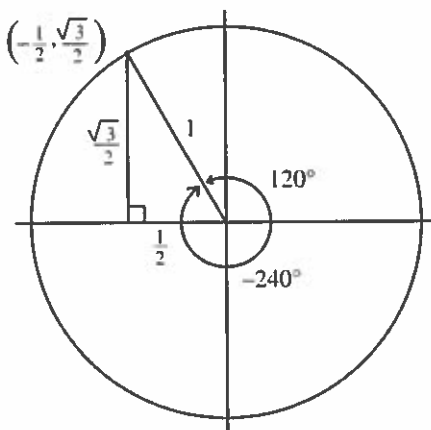


Figure 10.11



## ► Practice

Find the point on the unit circle that corresponds to the following angles.

9.  $-60^\circ$

10.  $\frac{2\pi}{3}$

11.  $420^\circ$

12.  $-120^\circ$

13.  $\frac{3\pi}{2}$

14.  $-\frac{2\pi}{3}$

15.  $180^\circ$

16.  $\frac{10\pi}{3}$

► Multiples of  $30^\circ$ 

If we flip the 30-60-90 triangle on its side, we can see that  $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$  corresponds to  $30^\circ$ , as shown in Figure 10.12.

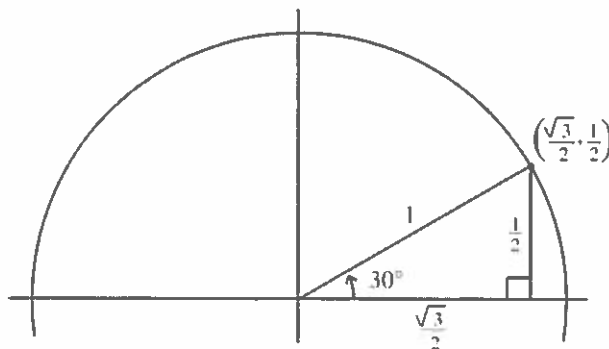


Figure 10.12

By flipping this triangle across the axes, we can find the points that correspond to  $150^\circ$  ( $\frac{5\pi}{6}$  radians),  $210^\circ$  ( $\frac{7\pi}{6}$  radians), and  $330^\circ$  ( $\frac{11\pi}{6}$  radians) (see Figure 10.13).

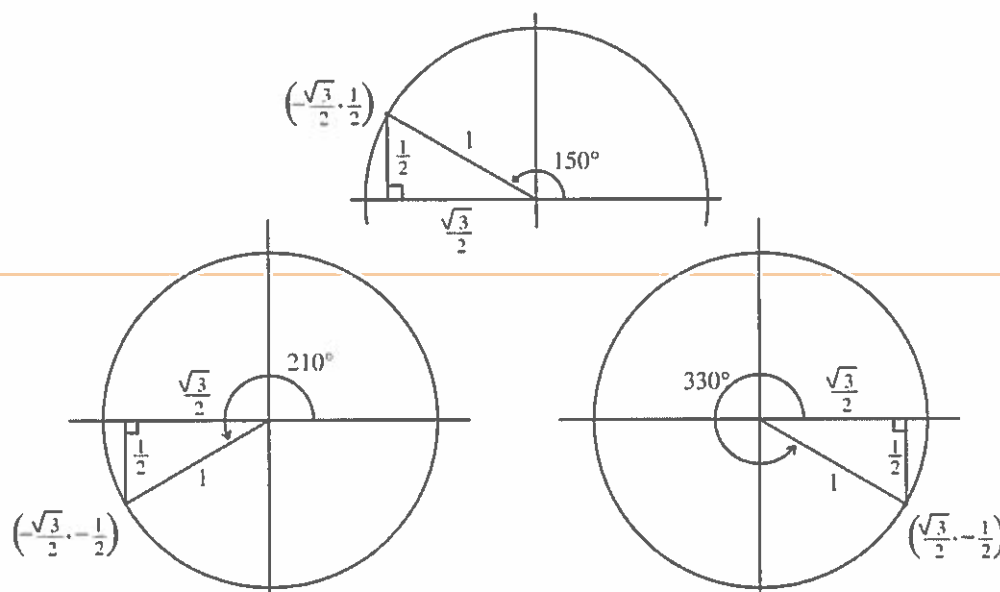


Figure 10.13

### ► Practice

Find the point on the unit circle that corresponds to the given angle.

17.  $\frac{5\pi}{6}$

18.  $-30^\circ$

19.  $\frac{7\pi}{6}$

20.  $390^\circ$

21.  $\frac{\pi}{3}$

22.  $-270^\circ$

23.  $570^\circ$

24.  $-\frac{5\pi}{6}$

► Multiples of  $45^\circ$ 

The  $45^\circ$  triangle used in the last lesson, shown in Figure 10.14, cannot fit inside the unit circle because the hypotenuse is too long. To get a similar triangle with hypotenuse 1, divide all the sides by  $\sqrt{2}$  (scale by a factor of  $\frac{1}{\sqrt{2}}$ , to get the triangle in Figure 10.15.

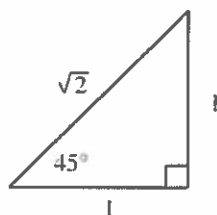


Figure 10.14

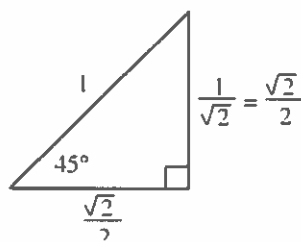


Figure 10.15

The point on the unit circle that corresponds to  $45^\circ$  is thus  $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ , as shown in Figure 10.16.

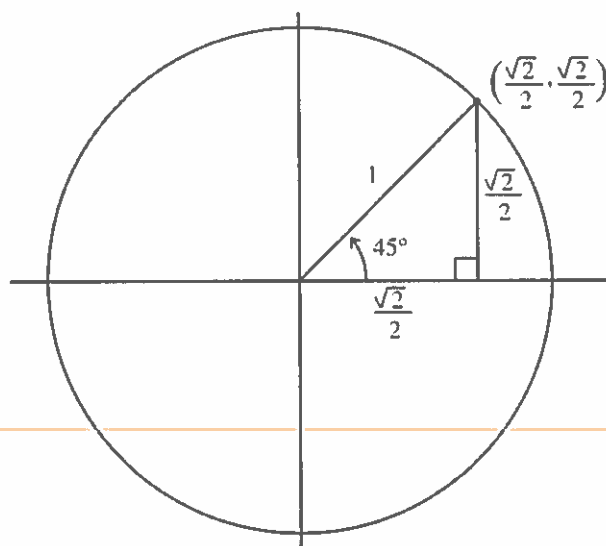


Figure 10.16

Figure 10.17 shows that the point  $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$  corresponds to  $135^\circ$ . Similarly, Figure 10.18 shows that  $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$  corresponds to  $225^\circ$  and  $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$  corresponds to  $315^\circ$ .

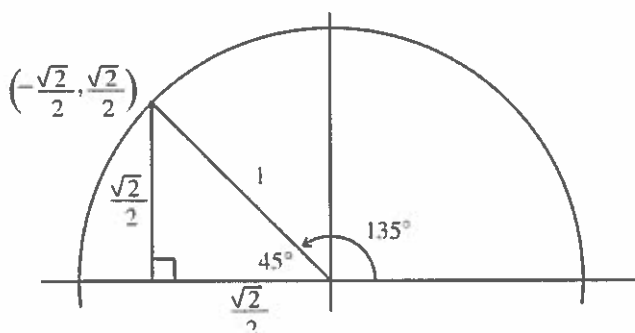


Figure 10.17

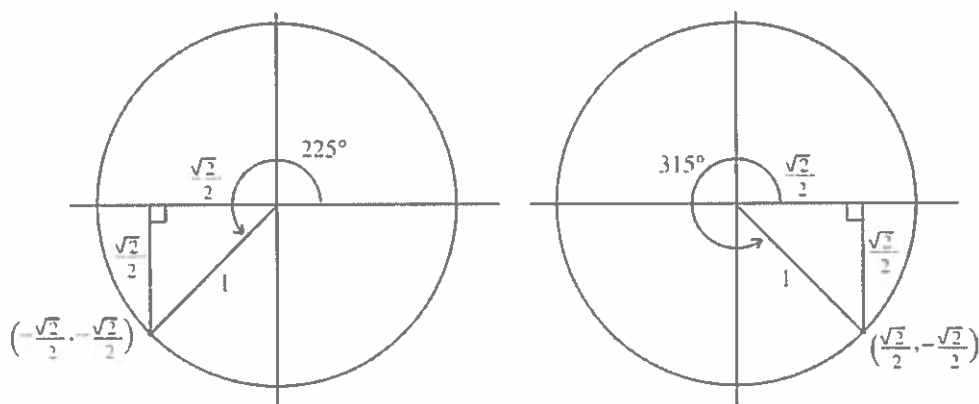


Figure 10.18

There are some tricks to finding the point of the unit circle that corresponds to a multiple of  $30^\circ$ ,  $45^\circ$ , and  $90^\circ$ .

For multiples of  $90^\circ = \frac{\pi}{2}$ , one coordinate is always 0, and the other is  $\pm 1$ .

For multiples of  $45^\circ = \frac{\pi}{4}$ , each coordinate is  $\pm \frac{\sqrt{2}}{2}$ .

For multiples of  $30^\circ = \frac{\pi}{6}$ , the smaller coordinate is  $\pm \frac{1}{2}$  and the longer is  $\pm \frac{\sqrt{3}}{2}$ .

A brief sketch of the angle usually suffices to indicate which coordinates are negative, and if one of the coordinates is longer.

### Example

What point of the unit circle corresponds to  $-\frac{5\pi}{3}$ ?

If we sketch the unit circle and count clockwise around by thirds of  $\pi$ , we find the approximate direction of this angle, shown in Figure 10.19.

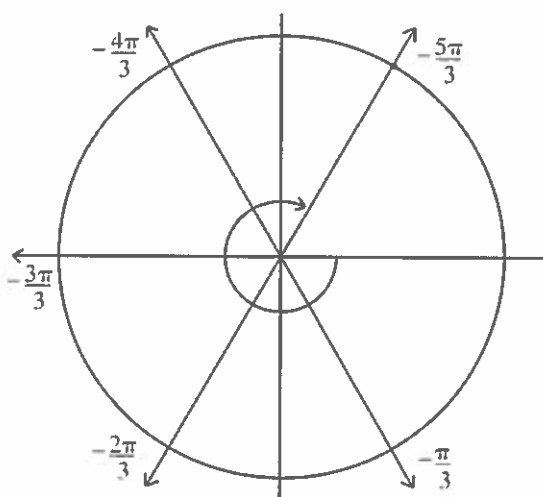


Figure 10.19

The  $y$ -coordinate looks larger than the  $x$ -coordinate, so the corresponding point is  $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ .

### Example

What point of the unit circle corresponds to  $\frac{3\pi}{4}$ ?

Here, we count counterclockwise by fourth of  $\pi$ , as in Figure 10.20.

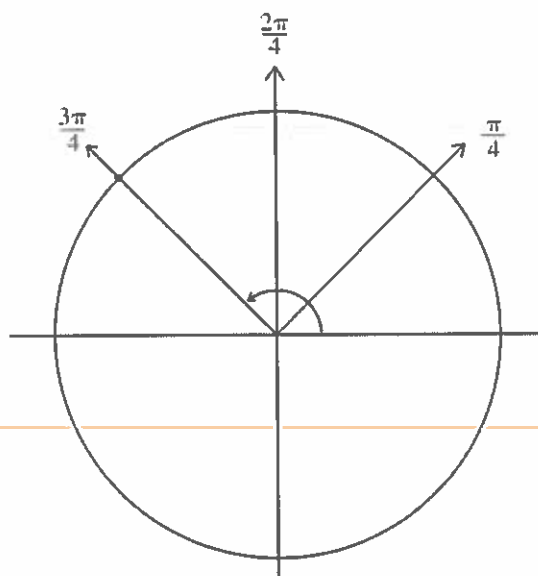


Figure 10.20

The two coordinates have the same length, so each is  $\pm \frac{\sqrt{2}}{2}$ . Here, the corresponding point is  $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ .

### ► Practice

Find the point of the unit circle that corresponds to the given angle.

25.  $\frac{3\pi}{4}$

26.  $405^\circ$

27.  $-45^\circ$

28.  $\frac{\pi}{6}$

29.  $810^\circ$

# L E S S O N

# 11



# Nice Angles

**W**

e now evaluate the trigonometric values of angles that are larger than  $90^\circ$  and less than  $0^\circ$ . The sine and cosine come directly from the  $y$ - and  $x$ -coordinates of the point on the unit circle, which correspond to the given angle.

The **nice angles** are the multiples of  $30^\circ$ ,  $45^\circ$ ,  $60^\circ$ , and  $90^\circ$   $\left(\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \text{ and } \frac{\pi}{2}\right)$ . We can find the point on the unit circle, which corresponds to any nice angle. The coordinates of this point are trigonometric values of the angle.

For example, suppose  $\theta$  is an angle between  $0^\circ$  and  $90^\circ$ . This angle must correspond to some point  $(x, y)$  on the unit circle. This means that there is a right triangle with hypotenuse 1, opposite side  $y$ , and adjacent side  $x$ , as shown in Figure 11.1.

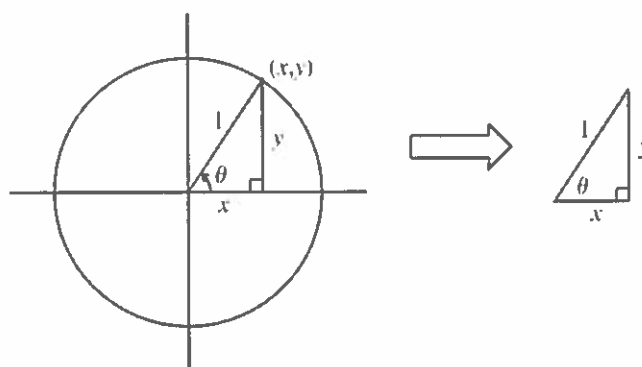


Figure 11.1

The sine of this angle is  $\sin(\theta) = \frac{y}{1} = y$ . The cosine of this angle is  $\cos(\theta) = \frac{x}{1} = x$ . In other words, the  $x$ -coordinate is the cosine and the  $y$ -coordinate is the sine, as shown in Figure 11.2. We use this to define the sine and cosine of every angle, even negative angles and angles greater than  $90^\circ$ .

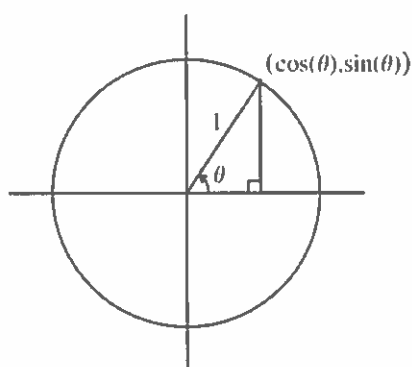


Figure 11.2

### Example

Find  $\cos(120^\circ)$ .

An angle of  $120^\circ$  is a little more ( $30^\circ$ ) than  $90^\circ$ . Thus, it looks like the angle in Figure 11.3. The  $y$ -coordinate is longer, so  $y = \pm \frac{\sqrt{3}}{2}$  and  $x = \pm \frac{1}{2}$ . Because the point is above and to the left of the origin, the point is  $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ . The  $x$ -coordinate is the cosine of the angle, so  $\cos(120^\circ) = -\frac{1}{2}$ .



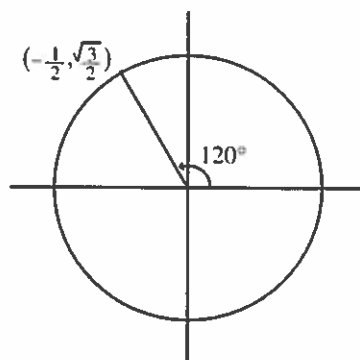


Figure 11.3

**Example**

Evaluate  $\sin(90^\circ)$ .

The point of the unit circle that corresponds to  $90^\circ$  is  $(0,1)$ , as shown in Figure 11.4.

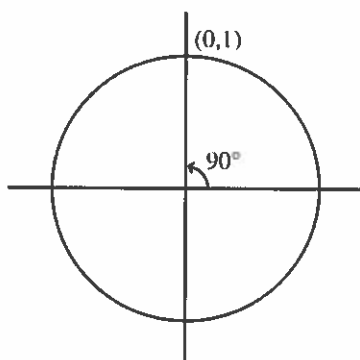


Figure 11.4

Because the sine is the same as the  $y$ -coordinate of the point,  $\sin(90^\circ) = 1$ .

**Example**

Find the sine and cosine of  $-\frac{\pi}{4}$ .

$$-\frac{\pi}{4} \text{ is the same as } -\frac{\pi}{4} \cdot \frac{180^\circ}{\pi} = -45^\circ.$$

This angle corresponds with the point  $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ , as shown in Figure 11.5.

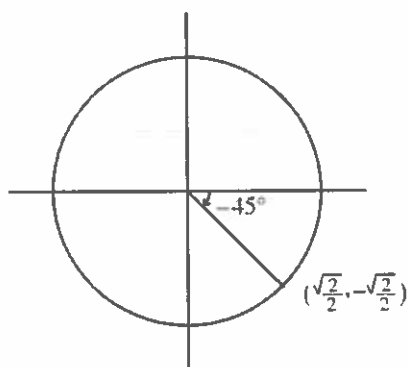


Figure 11.5

The  $y$ -coordinate is the sine and the  $x$ -coordinate is the cosine, so

$$\sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2} \text{ and } \cos\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

### ► Practice

Find the sine and cosine of each angle.

1.  $45^\circ$
2.  $180^\circ$
3.  $\frac{\pi}{3}$
4.  $\frac{\pi}{2}$
5.  $135^\circ$
6.  $\frac{5\pi}{4}$
7.  $390^\circ$
8.  $-60^\circ$
9.  $\frac{7\pi}{4}$

10.  $\frac{5\pi}{6}$

11.  $0^\circ$

12.  $\pi$

13.  $-\frac{\pi}{6}$

14.  $-\frac{2\pi}{3}$

15.  $330^\circ$

16.  $\frac{7\pi}{6}$

### ► Other Trigonometric Functions of Nice Angles

Once we know the sine and cosine of an angle  $\theta$ , we can put them together to evaluate any other trigonometric function. We use the following formulas:

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

$$\sec(\theta) = \frac{1}{\cos(\theta)}$$

$$\csc(\theta) = \frac{1}{\sin(\theta)}$$

$$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$

#### Example

What is  $\sec(135^\circ)$ ?

The angle  $135^\circ$  corresponds to the point  $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ , as shown in Figure 11.6.

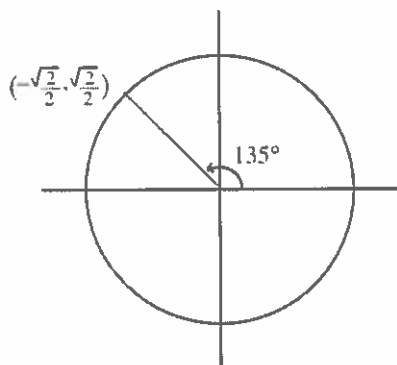


Figure 11.6

Thus,

$$\sin(135^\circ) = \frac{\sqrt{2}}{2}, \cos(135^\circ) = -\frac{\sqrt{2}}{2}$$

and

$$\sec(135^\circ) = \frac{1}{\cos(135^\circ)} = \frac{1}{-\frac{\sqrt{2}}{2}} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

### Example

Find  $\cot\left(-\frac{5\pi}{6}\right)$ .

$$-\frac{5\pi}{6} \text{ is the same as } -\frac{5\pi}{6} \cdot \frac{180^\circ}{\pi} = -150^\circ$$

which corresponds to  $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ , as shown in Figure 11.7.

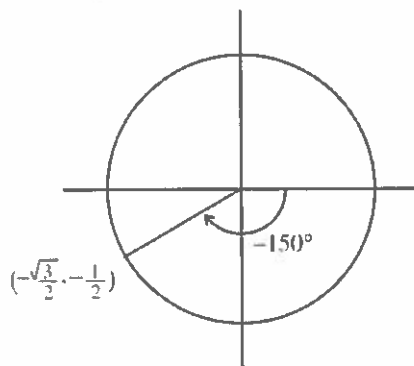


Figure 11.7

This means that  $\sin\left(-\frac{5\pi}{6}\right) = -\frac{1}{2}$  and  $\cos\left(-\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$ , so

$$\cot\left(-\frac{5\pi}{6}\right) = \frac{\cos\left(-\frac{5\pi}{6}\right)}{\sin\left(-\frac{5\pi}{6}\right)} = \frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \sqrt{3}$$

**Example**

Find  $\tan(90^\circ)$ . The point (0,1) on the unit circle corresponds to the  $90^\circ$  angle. Thus  $\cos(90^\circ) = 0$  and  $\sin(90^\circ) = 1$ . The tangent of  $90^\circ$  should be

$$\tan(90^\circ) = \frac{\sin(90^\circ)}{\cos(90^\circ)} = \frac{1}{0}$$

except that division by zero is never allowed.

The only way out of this mess is to declare that  $\tan(90^\circ)$  is undefined. Similarly,  $\sec(90^\circ)$  is also not defined because it would involve division by  $\cos(90^\circ) = 0$  if it did exist. At each multiple of  $90^\circ$  (such as  $90^\circ$ ,  $180^\circ$ ,  $270^\circ$ ,  $0^\circ$ ,  $-90^\circ$ ,  $-180^\circ$ , and so on), there will be undefined functions because one of the coordinates (sine or cosine) will be zero.

**► Practice**

Evaluate each of the following trigonometric functions.

17.  $\sec(60^\circ)$

18.  $\csc(-30^\circ)$

19.  $\sec\left(\frac{\pi}{6}\right)$

20.  $\tan(0)$

21.  $\cot(150^\circ)$

22.  $\csc(120^\circ)$

23.  $\sec(-45^\circ)$

24.  $\tan\left(\frac{\pi}{4}\right)$

25.  $\cot\left(\frac{\pi}{6}\right)$

26.  $\tan(240^\circ)$

27.  $\tan\left(-\frac{\pi}{3}\right)$

28.  $\cot(180^\circ)$

29.  $\csc(225^\circ)$

### ► Graphing Sine, Cosine, and Tangent

Now that we know the sine of many angles, we can use this to sketch the graph of  $y = \sin(x)$  shown in Figure 11.8.

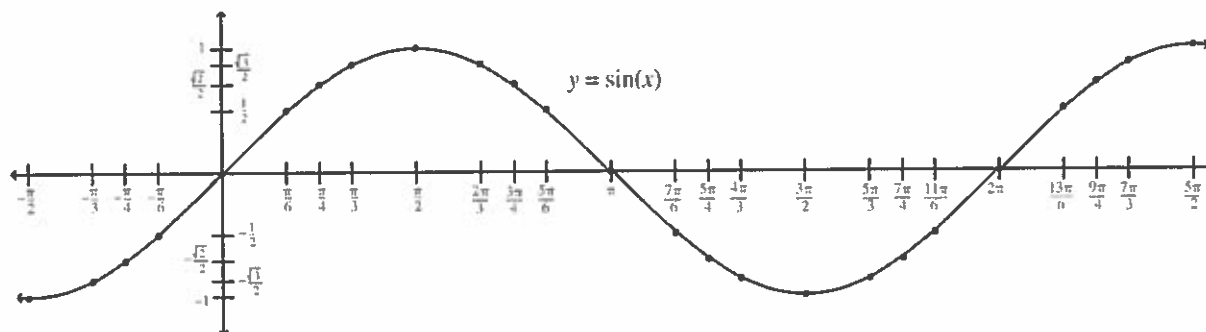


Figure 11.8

This graph oscillates up and down forever between 1 and  $-1$ . As the angles on the  $x$ -axis increase, the graph gives the height of the corresponding point on the unit circle, rising up, then down, and then repeating.

The graph of  $y = \cos(x)$ , shown in Figure 11.9, is almost the same, but shifted over by  $90^\circ$ , or  $\pi$ , radians. This is because the side-to-side motion of a point going around the unit circle follows the same pattern as its up-and-down motion.

The functions  $\sin(x)$  and  $\cos(x)$  are said to be **periodic** with period  $2\pi$  because they repeat every  $2\pi = 360^\circ$  (every full circle).

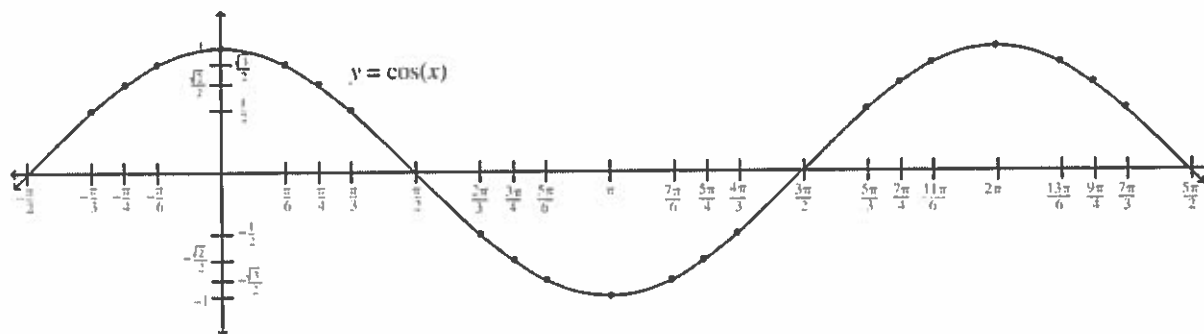


Figure 11.9

The graph of  $y = \tan(x)$ , shown in Figure 11.10, is more curious. Because  $\tan(x) = \frac{\sin(x)}{\cos(x)}$ , it is undefined everywhere  $\cos(x) = 0$ . When  $\cos(x)$  gets close to zero, the values of  $\tan(x)$  get really big and approach **vertical asymptotes**, at  $x = \frac{\pi}{2}$ ,  $x = \frac{3\pi}{2}$ , and at every other place that could be described as  $x = \frac{\pi}{2} + \pi \cdot k$  where  $k$  is an integer.

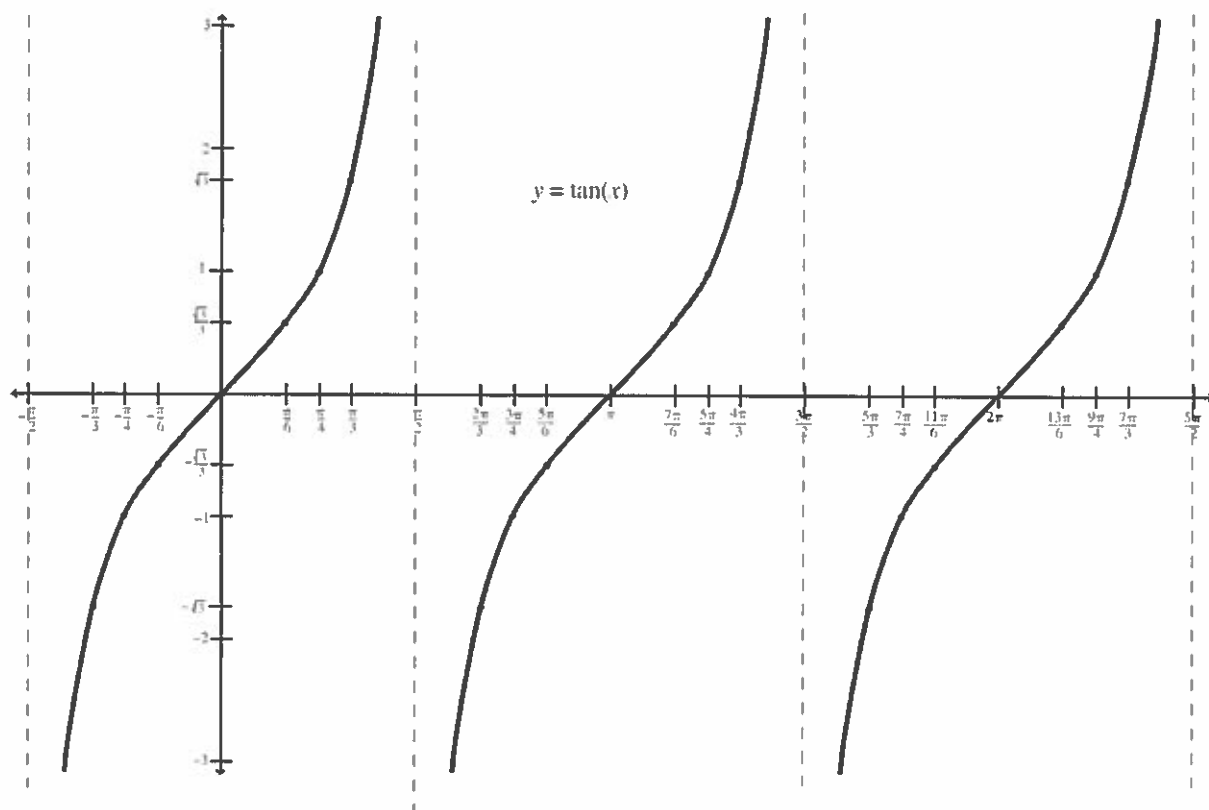


Figure 11.10

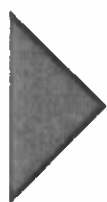
The tangent function is also periodic, but the period is  $\pi$ .





L E S S O N

# 12



## Calculator Trigonometry

**T**his lesson contains the essence of trigonometry but without the explanations. With a calculator, you can find any side of a right triangle if you know one side and an angle. The only disadvantage to using a calculator is that you get only approximate answers. However, several decimal places of accuracy will be close enough in any practical situation. If you don't have a calculator, it's a good idea to get one that has buttons that say SIN, COS, and TAN somewhere up near the top. A graphing calculator is unnecessary. A perfectly decent scientific calculator can be bought for just more than \$10.

Suppose we have the right triangle in Figure 12.1. The hypotenuse has length  $H$ , one of the angles has measure  $\theta$ , the side adjacent to  $\theta$  has length  $A$ , and the opposite side has length  $O$ .

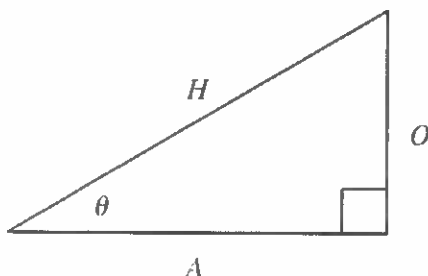


Figure 12.1

The Pythagorean theorem states that

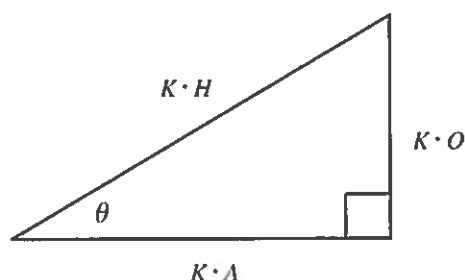
$$A^2 + O^2 = H^2$$

For example, if  $A = 8$  inches long and  $O = 5$  inches long, then

$$H = \sqrt{8^2 + 5^2} = \sqrt{64 + 25} = \sqrt{89} \approx 9.43398 \text{ inches long}$$

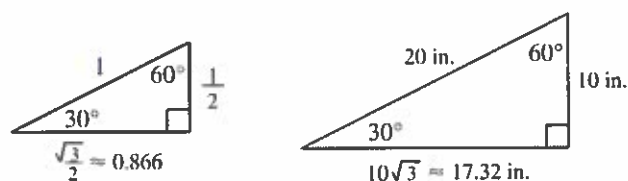
On an inexpensive calculator, you estimate  $\sqrt{89}$  by hitting “8,” then “9,” and then the square root button “ $\sqrt{\phantom{x}}$ ” or “ $\sqrt{x}$ .” On fancier calculators, like graphing calculators, you have to hit the square root button first, then type “89,” and then close up with a right parenthesis “)” so that the screen reads “ $\sqrt{(89)}$ ” before hitting “Enter.”

If you had any other right triangle with angle  $\theta$ , then it would be a scaled version (either shrunk or enlarged) with each side some number  $K$  times larger, as shown in Figure 12.2.



**Figure 12.2**

For example, suppose we had a  $30^\circ$ - $60^\circ$ - $90^\circ$  triangle with a hypotenuse of length 20 inches. It would have to be a scale multiple  $K$  of the classic  $30^\circ$ - $60^\circ$ - $90^\circ$  triangle, shown in Figure 12.3.



**Figure 12.3**

Each side of the larger triangle is exactly 20 times bigger.

The essence of trigonometry is that the ratio of one side divided by another in Figure 12.1 depends only on the angle  $\theta$ . These ratios are the trigonometric values of the angle  $\theta$ , called sine, cosine, tangent, secant, cosecant, and cotangent.

$$\sin(\theta) = \frac{O}{H}$$

$$\cos(\theta) = \frac{A}{H}$$

$$\tan(\theta) = \frac{O}{A}$$

$$\sec(\theta) = \frac{H}{A}$$

$$\csc(\theta) = \frac{H}{O}$$

$$\cot(\theta) = \frac{A}{O}$$

To evaluate these ratios for a given angle  $\theta$ , all you need is a calculator. The most important detail of all is to make sure that the calculator is set to “DEG” if the angle  $\theta$  is given in degrees, like  $34^\circ$  or  $14.2^\circ$ . If  $\theta$  is measured in radians, like  $\frac{\pi}{4}$  or  $\frac{2\pi}{3}$ , then the calculator needs to be set to “RAD.” Inexpensive calculators have a “DEG” button, which cycles between “DEG,” “RAD,” and “GRAD” (a type of angle measurement where a right angle measures 100 gradians). Usually one of these three is displayed in small letters somewhere on the screen. Fancier calculators sometimes require you to adjust between degrees and radians on an options or preferences menu. Most calculators start in degrees mode, but don’t assume this.

### Tip

If you don’t like radians, just convert to degrees by multiplying by  $\frac{180^\circ}{\pi}$ .

On an inexpensive calculator, type “30” and then hit the “SIN” button. If the screen reads 0.5, then the calculator is in degrees mode. If the screen reads  $-0.988031624$ , then the calculator is in radians mode. On a fancier calculator, you usually have to hit the “SIN” button first, type “30,” then close the right parentheses “)” and hit “Enter.”

### Example

Estimate  $\cos(42^\circ)$  to three decimal places.

$$\cos(42^\circ) \approx 0.743$$

(Remember that “ $\approx$ ” means “approximately equal to.”)

### Example

Estimate  $\tan\left(\frac{5\pi}{7}\right)$  to three decimal places.

$$\tan\left(\frac{5\pi}{7}\right) \approx -1.254$$

### ► Practice

Estimate each of the trigonometric values to three decimal places with a calculator.

1.  $\sin(50^\circ)$

2.  $\cos(30^\circ)$

3.  $\tan(85^\circ)$

4.  $\sin(315^\circ)$

5.  $\cos(-80^\circ)$

6.  $\tan(14^\circ)$

7.  $\sin\left(\frac{\pi}{5}\right)$

8.  $\cos(\pi)$

9.  $\tan\left(\frac{\pi}{4}\right)$

10.  $\sin\left(-\frac{\pi}{10}\right)$

11.  $\cos(8.63)$

12.  $\tan\left(\frac{\pi}{3}\right)$

### ► Secant, Cosecant, and Cotangent on a Calculator

Sine, cosine, and tangent are the only trigonometric values that we need to solve any trigonometry word problem. The others are merely reciprocals (flips) of these:

$$\sec(\theta) = \frac{1}{\cos(\theta)}$$

$$\csc(\theta) = \frac{1}{\sin(\theta)}$$

$$\cot(\theta) = \frac{1}{\tan(\theta)}$$

In fact, inexpensive calculators often don't have "SEC," "CSC," or "COT" buttons, so you have to use these identities.

#### Example

Estimate  $\sec(48^\circ)$  to three decimal places with a calculator.

$$\sec(48^\circ) = \frac{1}{\cos(48^\circ)} \approx \frac{1}{0.669130606} \approx 1.494$$

#### Example

Estimate  $\cot\left(\frac{\pi}{8}\right)$  to three decimal places with a calculator.

$$\cot\left(\frac{\pi}{8}\right) = \frac{1}{\tan\left(\frac{\pi}{8}\right)} \approx \frac{1}{0.414213562} \approx 2.414$$

### ► Practice

Evaluate each trigonometric value to three decimal places with a calculator.

13.  $\sec(18^\circ)$

14.  $\csc(412^\circ)$

15.  $\cot(-23.1^\circ)$

16.  $\sec\left(\frac{\pi}{18}\right)$

17.  $\csc\left(-\frac{3\pi}{4}\right)$

18.  $\cot(2.61)$

## ► Finding Sides of a Right Triangle

With a calculator, we can accurately estimate the length of any side of a right triangle if we know one side and an angle.

### Example

What is the length of  $x$  of the triangle in Figure 12.4?

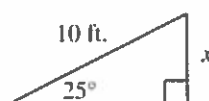


Figure 12.4

The two sides in concern are the side opposite the  $25^\circ$  angle,  $O = x$ , and the hypotenuse  $H = 10$  feet. The sine relates these two sides:

$$\sin(25^\circ) = \frac{O}{H} = \frac{x}{10}$$

Using a calculator, we estimate  $\sin(25^\circ) \approx 0.423 \approx \frac{x}{10}$  and solve for  $x \approx 4.23$  feet.

### Example

Estimate the length  $x$  of the triangle in Figure 12.5.

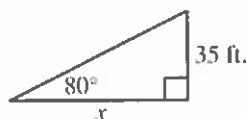


Figure 12.5

Here,  $O = 35$  feet and  $A = x$ , so

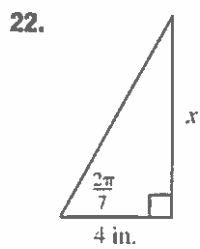
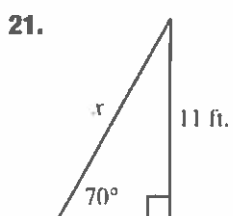
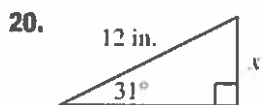
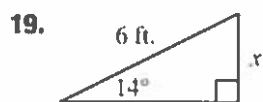
$$\tan(80^\circ) = \frac{O}{A} = \frac{35}{x}$$

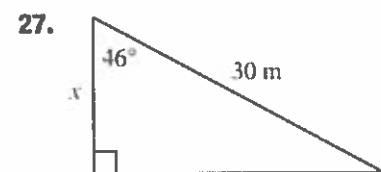
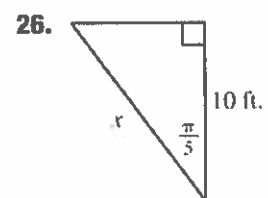
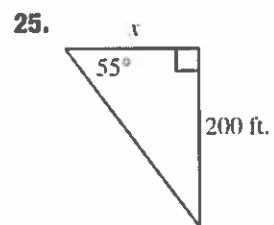
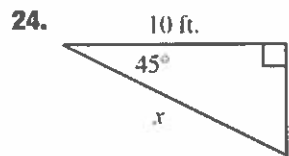
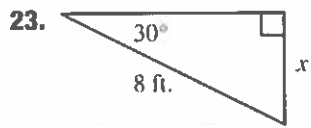
$$5.671 \approx \frac{35}{x}$$

$$x \approx \frac{35}{5.671} \approx 6.172 \text{ feet}$$

### ► Practice

Use a calculator to estimate the length  $x$  in each triangle.







L E S S O N

# 13



## Finding Lengths

In this lesson, we use the basic trigonometric functions to solve word problems. In these problems, we use an angle and a distance to find the other distances around a right triangle.

It is fortunate for us that right triangles appear in many situations. Walls, poles, and buildings are built at right angles to the ground. Trees grow upward at right angles to the ground (usually). Nearly everything that human beings make involves straight lines and right angles. Even where no lines exist, such as a plane flying in the air, we can imagine a line running straight down to the ground, giving the altitude and making a right angle with the ground.

The trick to solving such problems is to sketch the right triangle. Usually, the ground forms one of the sides. The right angle is then formed by the wall, height, pole, tree, or something else that rises straight up. The length that is given and the one that is sought will be either  $O$  and  $H$ ,  $A$  and  $H$ , or  $O$  and  $A$ . The angle  $\theta$  that is given will thus be  $\sin(\theta) = \frac{O}{H}$ ,  $\cos(\theta) = \frac{A}{H}$ , or  $\tan(\theta) = \frac{O}{A}$ . Write out this equation, then solve for the desired length, using a calculator to estimate the trigonometric function.

**Example**

Suppose a 20-foot ladder leans against a wall. If it makes a  $75^\circ$  angle with the ground, how high up does it reach on the wall?

We suppose that the wall rises up at a right angle to the ground, and so we sketch Figure 13.1.

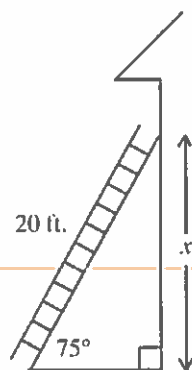


Figure 13.1

The side that is given is the hypotenuse  $H = 20$  feet, and the side that we want to know is the opposite  $O$ , which we label  $x$ . Thus, we use the sine function.

$$\sin(\theta) = \frac{O}{H}$$

$$\sin(75^\circ) = \frac{x}{20}$$

$$x = 20 \sin(75^\circ) \approx 19.3 \text{ feet}$$

Thus, the ladder will rest at a spot 19.3 feet up the wall. Note: Because there are no absolute rules about where to round off, the answer could be 19 feet, 19.32 feet, 19.319 feet, or anything similar.

**Example**

The manufacturer of a ladder decides that the ladder should never be used at more than an  $80^\circ$  angle to the ground. When a 35-foot ladder is leaned against a wall, how close can the bottom of the ladder be from the wall?

Here, we sketch the triangle in Figure 13.2. We want to know the length of adjacent side  $A$  when the hypotenuse is  $H = 35$  feet.

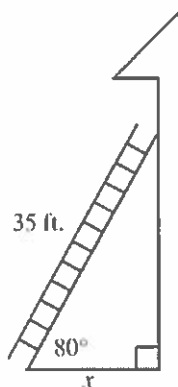


Figure 13.2

$$\cos(\theta) = \frac{A}{H}$$

$$\cos(80^\circ) = \frac{x}{35}$$

$$x = 35 \cos(80^\circ) \approx 6.08 \text{ feet}$$

The manufacturer should insist that people keep the base of the ladder 6.08 feet (or, just more than 6 feet) from the base of the wall to avoid accidents.

### Example

Suppose a painter wants a ladder to rest against a spot 40 feet up a wall. If the ladder will make a  $72^\circ$  angle with the ground, how long must the ladder be?

We sketch the triangle in Figure 13.3. Here, the side opposite the angle is  $O = 40$  feet long and the hypotenuse  $H$  is the unknown length.

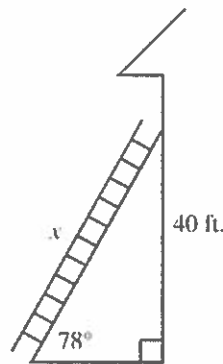


Figure 13.3

$$\sin(\theta) = \frac{O}{H}$$

$$\sin(72^\circ) = \frac{40}{x}$$

$$x = \frac{40}{\sin(72^\circ)} \approx 42 \text{ feet}$$

The ladder must be 42 feet long.

**Example**

When a person stands 100 feet away from a skyscraper, the top of the building appears to be at an  $82^\circ$  angle from horizontal. How tall is the building?

We sketch the triangle in Figure 13.4. Here, we know that the length of the adjacent side is  $A = 100$  feet, but we want to know the length of the opposite side  $O$ .

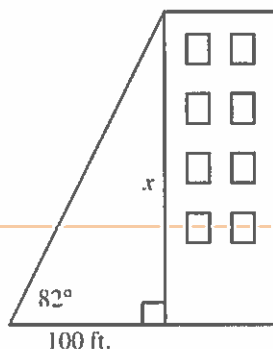


Figure 13.4

$$\tan(\theta) = \frac{O}{A}$$

$$\tan(82^\circ) = \frac{x}{100}$$

$$x = 100 \tan(82^\circ) \approx 712 \text{ feet}$$

The building is approximately 712 feet tall.

**Example**

A window is 45 feet off the ground. A person looks out and sees a car on the road. If the car appears to be  $10^\circ$  below horizontal, how far away is the car from the building?

We sketch Figure 13.5. There are two ways to look at this situation. We could use the top triangle, with a  $10^\circ$  angle, an opposite side of  $O = 45$  feet, and an unknown adjacent side  $A$ . In this case,

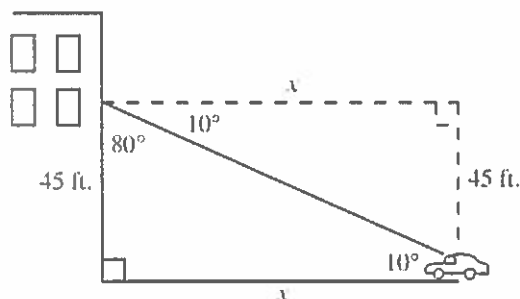


Figure 13.5

$$\tan(\theta) = \frac{O}{A}$$

$$\tan(10^\circ) = \frac{45}{x}$$

$$x = \frac{45}{\tan(10^\circ)} \approx 255 \text{ feet}$$

If we use the lower triangle, then the angle at the window has to be  $80^\circ$  because the height of the wall makes a right angle ( $90^\circ$ ) with the horizontal. If the diagonal is  $10^\circ$  down from horizontal, it must also be  $80^\circ$  up from the wall. Now, the unknown side is the opposite side  $O$ , and the known side is the adjacent  $A = 45$  feet.

$$\tan(\theta) = \frac{O}{A}$$

$$\tan(80^\circ) = \frac{x}{45}$$

$$x = 45 \tan(80^\circ) \approx 255 \text{ feet}$$

Both methods reach the same conclusion: that the car is about 255 feet away from the building.

In the last problem, we wanted to know the distance from the car to the building along the ground. If we had wanted to know the direct distance through the air from the person in the window to the car, then we would have used the hypotenuse  $H$  as the unknown  $x$ . Usually, distances are measured along the ground.

### Example

A 5-foot-tall photographer wants to view a 20-foot-tall tree so that the top of the tree appears within  $30^\circ$  of horizontal. How far away from the tree must the photographer stand?

We sketch Figure 13.6. Because the photographer is 5 feet tall, we imagine that the camera, aimed horizontally, points at a spot on the tree about 5 feet off the ground. Only the top 15 feet of the tree must be fit into the  $30^\circ$  angle. Thus, the opposite side has length  $O = 15$  feet and the adjacent side  $A$  is what we want to know.

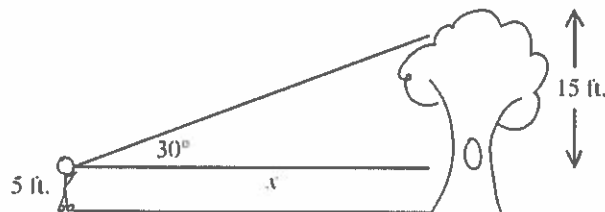


Figure 13.6

$$\tan(\theta) = \frac{O}{A}$$

$$\tan(30^\circ) = \frac{15}{x}$$

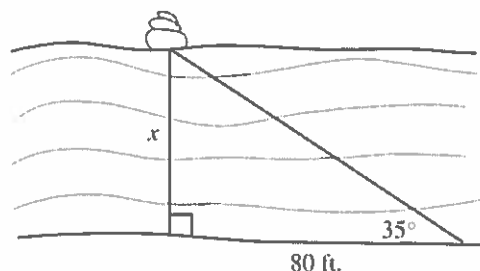
$$x = \frac{15}{\tan(30^\circ)} \approx 26 \text{ feet}$$

The photographer should stand back about 26 feet.

**► Practice**

1. A 25-foot ladder leans against a wall. If it makes an angle of  $72^\circ$  with the ground, how high up the wall does it reach?
2. A 40-foot ladder leans against a tree. If the angle formed by the ladder and the ground measures  $81^\circ$ , how high up in the tree does the ladder reach?
3. A 30-foot ladder comes with instructions that say that it should never make more than a  $75^\circ$  angle with the ground. How far should the base of the ladder be kept from the base of a wall it is leaned against?
4. A 50-foot wire is pulled tight from the top of a pole to the ground. If it makes a  $53^\circ$  angle with the ground, how tall is the pole?
5. A wire will be attached from the top of a 20-foot telephone pole to the ground. If it should make a  $60^\circ$  angle with the ground, how long should the wire be?
6. A wire runs from the top of a radio tower to a point 25 feet from the base of the tower. If the wire makes an  $81^\circ$  angle with the ground, how tall is the tower?
7. A person at the top of a 100-foot tower sees a bear in the distance. If the bear appears to be  $5^\circ$  below the horizontal, how far is it from the base of the tower?
8. The pilot of an airplane sees a target at  $15^\circ$  below the horizontal. If the plane is 20,000 feet off the ground, how far away is the target (directly along the diagonal)?
9. The top of a tree appears to be  $25^\circ$  above the horizontal when viewed from 60 feet away. How tall is the tree?
10. A mountaintop appears  $10^\circ$  above the horizontal when a person views it from 5 miles away. How much higher is the mountain than where the person is standing?
11. A security camera is installed 20 feet from the wall of a museum. If the camera can rotate  $40^\circ$  to the left and  $40^\circ$  to the right, how many feet of wall can it observe?
12. A 6-foot-tall photographer wants to view a 50-foot-tall building so that it is no more than  $35^\circ$  from the horizontal. How far back should the photographer stand?
13. An airplane that is cruising at 30,000 feet appears to be  $50^\circ$  above the horizontal. How far away is the airplane (straight through the air)?

- 14.** A hiker sees a stack of rocks directly across a river. After he walks 80 feet along the riverside, the rocks appear at a  $35^\circ$  angle (see Figure 13.7). How wide is the river?



**Figure 13.7**

- 15.** When the sun is  $25^\circ$  above the horizon, a tree casts a 40-foot shadow. How tall is the tree?
- 16.** A person holds a laser pointer 3 feet off the ground. If it is aimed at  $4^\circ$  below horizontal, how far will the point of light be from where the person is standing?
- 17.** A hot air balloon is 2,000 feet off the ground. The destination can be seen at  $15^\circ$  below the horizontal. How far away is the destination (measured along the ground)?
- 18.** A 6-foot-long piece of wood leans against a wall. If it makes an angle of  $15^\circ$  with the wall, how far is the bottom of the board from the wall?
- 19.** A ramp runs at a  $20^\circ$  angle up to a height of 5 feet. How long is the diagonal of the ramp?
- 20.** A road descends from the top of a mountain at a  $5^\circ$  angle. If the road is 8 miles long to the bottom, how tall is the mountain?
- 21.** A 50-foot rope is tied to a window. A person on the ground takes the end of the rope and walks until it is pulled straight. At this point, the rope makes a  $37^\circ$  angle with the ground. How high up is the window?
- 22.** When the sun is  $28^\circ$  above the horizon, a tall rock mesa casts a shadow that is 400 feet long. How tall is the mesa?
- 23.** A ramp must be built at a  $14.5^\circ$  angle to reach a point 4.2 feet off the ground. How long must the ramp be (along the ground)?
- 24.** If a 10-foot pole is leaned against a wall at a  $45^\circ$  angle, how high up will it reach?
- 25.** Two buildings, 100 feet apart, face each other. A person in one building looks up at a  $58^\circ$  angle (from the horizontal) to a person in the other building. What is the diagonal distance between them?





L E S S O N

# 14



## Inverse Functions

**I**n this lesson, we introduce inverse trigonometric functions. With these, we are able to find the angles of right triangles whose sides are known.

When we plug an angle  $\theta$  into a trigonometric function like sine, out comes a ratio  $r$  of lengths. The **inverse sine** (written either  $\sin^{-1}$  or  $\arcsin$ ) is a function that does the exact opposite. If you plug in a ratio  $r$  of sides, out will come the corresponding angle  $\theta$ .

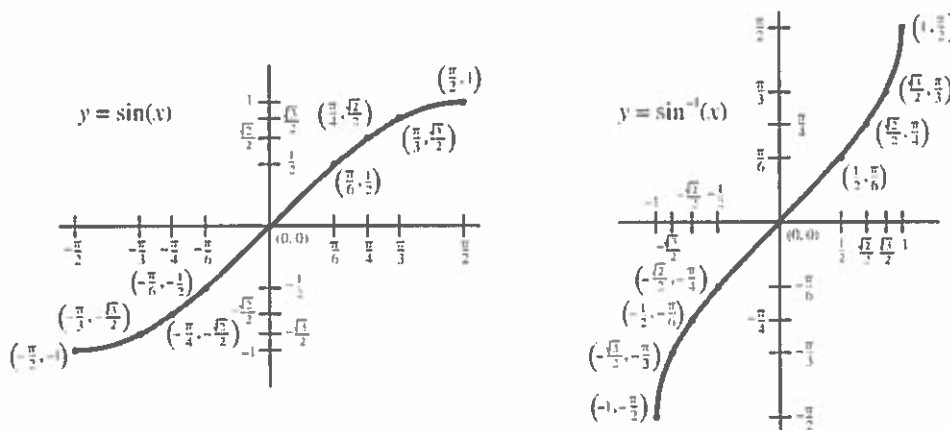
For example,  $\sin(30^\circ) = \frac{1}{2}$ , so  $\sin^{-1}\left(\frac{1}{2}\right) = 30^\circ$ . Similarly,  $\sin(90^\circ) = 1$ , so  $\sin^{-1}(1) = 90^\circ$ . In general, if  $\sin(\theta) = r$ , then  $\sin^{-1}(r) = \theta$ .

## Tip

The  $-1$  in  $\sin^{-1}$  indicates that this function is the opposite of sine. This does not mean that  $\sin^{-1}(x) = \frac{1}{\sin(x)}$ . The opposite of multiplying by 5 is multiplying by  $5^{-1} = \frac{1}{5}$ . However, taking the sine of an angle is more complicated than multiplication, so it cannot be undone merely by putting the sine in the denominator. In a word,  $\sin^{-1}(x)$  is different from  $\csc(x)$ .

Said another way,  $\sin^{-1}(r) =$  the angle  $\theta$  for which  $\sin(\theta) = r$ .

The one difficulty that arises is that several angles can have the same ratio, for example,  $\sin(45^\circ) = \frac{\sqrt{2}}{2}$  and  $\sin(135^\circ) = \frac{\sqrt{2}}{2}$ . What should  $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$  be? It cannot output two things. To avoid this problem, it has been decided that only angles  $-90^\circ \leq \theta \leq 90^\circ$   $\left(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\right)$  can come out of the  $\sin^{-1}$  function. Because  $-90^\circ \leq 45^\circ \leq 90^\circ$  (and  $135^\circ$  is not), we have  $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = 45^\circ$ . It might help to see the graph of  $y = \sin(x)$  on  $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$  and  $y = \sin^{-1}(x)$  in Figure 14.1.



**Figure 14.1**

Notice that the second graph is related to the first by reversing the roles of  $x$  and  $y$ . In general, the inverse function is found in this way, by reversing the  $x$ - and  $y$ -axes—in other words, “flipping” the graph about the line  $y = x$ .

There are also inverse functions to cosine and tangent. The inverse to  $\cos(\theta) = r$  is  $\cos^{-1}(r) = \theta$ , where the angle  $\theta$  must be  $0^\circ \leq \theta \leq 180^\circ$  or  $0 \leq \theta \leq \pi$ . The inverse to  $\tan(\theta) = r$  is  $\tan^{-1}(r) = \theta$ , where the angle  $\theta$  must be  $-90^\circ < \theta < 90^\circ$  or  $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ .

**Example**

Evaluate  $\cos^{-1}\left(\frac{1}{2}\right)$ .

$$\cos^{-1}\left(\frac{1}{2}\right) = 60^\circ = \frac{\pi}{3}, \text{ because } \cos(60^\circ) = \frac{1}{2} \text{ and } 0 \leq 60^\circ \leq 180^\circ$$

**Example**

Evaluate  $\tan^{-1}(1)$ .

$$\tan^{-1}(1) = 45^\circ = \frac{\pi}{4}, \text{ because } \tan(45^\circ) = 1 \text{ and } -90^\circ < 45^\circ < 90^\circ$$

**Example**

Evaluate  $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ .

$$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -60^\circ = -\frac{\pi}{3}, \text{ because } \sin(-60^\circ) = -\frac{\sqrt{3}}{2} \text{ and } -90^\circ \leq -60^\circ \leq 90^\circ$$

**► Practice**

Evaluate each of the following inverse trigonometric functions.

1.  $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

2.  $\tan^{-1}(\sqrt{3})$

3.  $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$

4.  $\cos^{-1}(-1)$

5.  $\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$

6.  $\sin^{-1}(0)$

7.  $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

8.  $\cos^{-1}\left(-\frac{1}{2}\right)$

9.  $\tan^{-1}(-\sqrt{3})$

10.  $\tan^{-1}(0)$

### ► Combining Functions with Inverses

We can plug an inverse trigonometric function into a regular function like  $\tan\left(\sin^{-1}\left(\frac{2}{5}\right)\right)$ . This asks, “What is the tangent of the angle whose sine is  $\frac{2}{5}$ ?” We have already dealt with such questions before, although now we use new notation.

Remember that  $\sin^{-1}\left(\frac{2}{5}\right)$  is the angle  $\theta$  with  $\sin(\theta) = \frac{2}{5}$ . Although we don’t know what  $\theta$  is exactly, we can draw a right triangle with angle  $\theta$ , as in Figure 14.2.

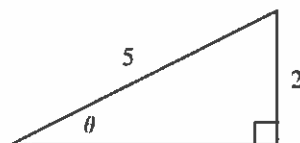


Figure 14.2

The opposite side divided by the hypotenuse must be  $\sin(\theta) = \frac{O}{H} = \frac{2}{5}$ . The easiest lengths to choose are  $O = 2$  and  $H = 5$ . Using the Pythagorean theorem, the adjacent side is  $A = \sqrt{25 - 4} = \sqrt{21}$ . Thus,

$$\tan\left(\sin^{-1}\left(\frac{2}{5}\right)\right) = \tan(\theta) = \frac{2}{\sqrt{21}} = \frac{2\sqrt{21}}{21}$$

### Example

Evaluate  $\cos(\tan^{-1}(4))$ .

$$\tan^{-1}(4) = \theta, \text{ where } \tan(\theta) = 4$$

The angle  $\theta$  is sketched in Figure 14.3.

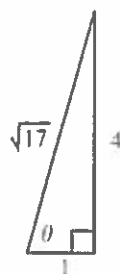


Figure 14.3

We choose  $O = 4$  and  $A = 1$  so that  $\tan(\theta) = \frac{O}{A} = \frac{4}{1} = 4$ . The hypotenuse  $H = \sqrt{16 + 1} = \sqrt{17}$  is found by the Pythagorean theorem. Thus,

$$\cos(\tan^{-1}(4)) = \cos(\theta) = \frac{1}{\sqrt{17}} = \frac{\sqrt{17}}{17}$$

### Example

Evaluate  $\sin(\sin^{-1}(0.7))$ .

$$\sin^{-1}(0.7) = \theta, \text{ where } \sin(\theta) = 0.7$$

The angle  $\theta$  is sketched in Figure 14.4.

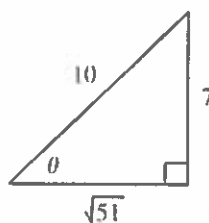


Figure 14.4

The opposite side is  $O = 7$  and the hypotenuse is  $H = 10$  so that  $\sin(\theta) = \frac{O}{H} = \frac{7}{10} = 0.7$ . The adjacent side is  $A = \sqrt{10^2 - 7^2} = \sqrt{51}$  by the Pythagorean theorem. Thus,

$$\sin(\sin^{-1}(0.7)) = \frac{7}{10} = 0.7$$

This problem could have been solved much more directly by pronouncing  $\sin(\sin^{-1}(0.7))$ . This reads, "What is the sine of the angle whose sine is 0.7?" Clearly, the sine of that angle is 0.7. It is in this sense that an inverse undoes a function. We might as well just cross out the  $\sin$  and  $\sin^{-1}$  from the original  $\sin(\sin^{-1}(0.7))$ , leaving behind just the 0.7.

### ► Practice

Evaluate each of the following.

11.  $\sin\left(\tan^{-1}\left(\frac{5}{2}\right)\right)$

12.  $\cos\left(\sin^{-1}\left(\frac{3}{4}\right)\right)$

13.  $\tan\left(\cos^{-1}\left(\frac{1}{7}\right)\right)$

14.  $\cos\left(\cos^{-1}\left(\frac{1}{4}\right)\right)$

15.  $\sin\left(\cos^{-1}\left(\frac{3}{5}\right)\right)$

16.  $\tan(\sin^{-1}(0.3))$

17.  $\tan(\tan^{-1}(9.2))$

18.  $\sin(\tan^{-1}(1.2))$

### ► Inverse Trigonometric Functions on a Calculator

A calculator can be used to estimate inverse trigonometric functions. For example, suppose you want to know what angle  $\theta$  has  $\sin(\theta) = \frac{1}{4}$ . Thus, we want to evaluate  $\theta = \sin^{-1}\left(\frac{1}{4}\right)$ . Usually  $\sin^{-1}$  is written in small print above the “SIN” button on a calculator. You need to first press “2nd” or “INV” and then press the “SIN” button. On an inexpensive scientific calculator, press “1,” “÷,” “4,” and then “=” to get 0.25 on the screen. Next, press “2nd” (or “INV”) and then “SIN.” The result will be around  $14.478^\circ$  if your calculator is in degrees mode and 0.253 if your calculator is in radians mode. On a fancier graphing calculator, press “INV” first, and then “SIN” so that the screen shows:

$$\sin^{-1}(\quad)$$

Next, type “1,” “÷,” “4,” “),” and “Enter” to get

$$\sin^{-1}(1/4) \approx 14.478^\circ$$

### Example

What is the measure of the angle  $x$  in Figure 14.5?

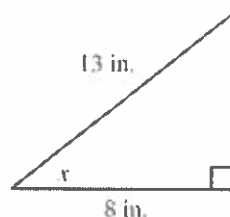


Figure 14.5

We know that  $\cos(x) = \frac{A}{H} = \frac{8}{13}$ . This means that  $x = \cos^{-1}\left(\frac{8}{13}\right)$ . With a calculator, this comes out to approximately  $x \approx 52^\circ$ , or 0.908 radians.

### Example

What is the measure of  $\theta$  in Figure 14.6?

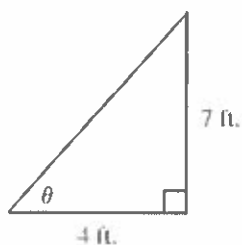


Figure 14.6

Here, we know that the opposite side is  $O = 7$  feet and the adjacent side is  $A = 4$  feet. Thus,  $\tan(\theta) = \frac{O}{A} = \frac{7}{4}$ . Equivalently,  $\theta = \tan^{-1}\left(\frac{7}{4}\right)$ . With a calculator, this comes out to  $\theta \approx 60.26^\circ$ , or 1.05 radians.

## ► Practice

Evaluate the following inverse trigonometric functions on a calculator.

19.  $\sin^{-1}(0.81)$

20.  $\cos^{-1}(-0.33)$

21.  $\tan^{-1}(4.2)$

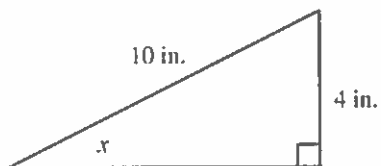
22.  $\sin^{-1}\left(-\frac{3}{5}\right)$

23.  $\cos^{-1}\left(\frac{11}{16}\right)$

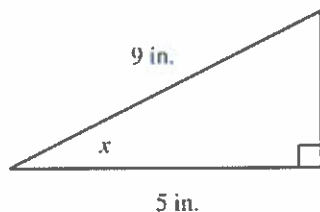
24.  $\tan^{-1}(-1.8)$

Find the measure of the angle  $x$  in each triangle.

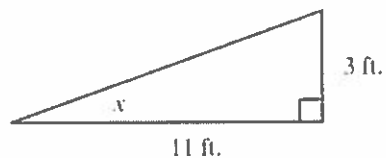
25.



26.



27.





L E S S O N

# 15



## Finding Angles

In this lesson, we explore a variety of word problems in which we must calculate the measure of an angle from a right triangle. With inverse trigonometric functions  $\sin^{-1}$ ,  $\cos^{-1}$ , and  $\tan^{-1}$ , we can find the angle of a right triangle given any two sides (see Figure 15.1).

$$\theta = \sin^{-1}\left(\frac{O}{H}\right)$$

$$\theta = \cos^{-1}\left(\frac{A}{H}\right)$$

$$\theta = \tan^{-1}\left(\frac{O}{A}\right)$$

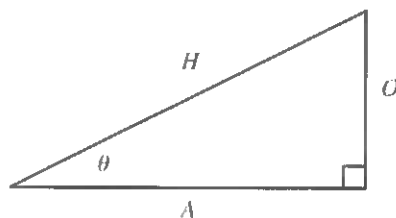


Figure 15.1

**Example**

A 20-foot board is leaning against a wall. If the board reaches 17 feet up the wall, what are the angles that the board makes with the floor and the wall?

The angle opposite the 17-foot wall will have  $\sin(\theta) = \frac{O}{H} = \frac{17}{20}$ , so  $\theta = \sin^{-1}\left(\frac{17}{20}\right)$ . A calculator shows this to be approximately  $58.2^\circ$ . The other angle must be the complement (make them add to  $90^\circ$ ),  $90 - 58.2 = 31.8^\circ$ . This is shown in Figure 15.2.

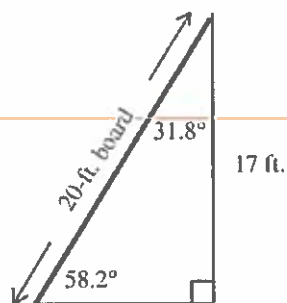


Figure 15.2

In the last example, we knew the length of the diagonal hypotenuse, and so we used sine. For most problems, we know the lengths of the two legs, and so we use tangent.

**Example**

Suppose you want to aim exactly 20 feet forward and 12 feet to the right. At what angle should you turn?

$$\theta = \tan^{-1}\left(\frac{O}{A}\right) = \tan^{-1}\left(\frac{12}{20}\right) \approx 30.964^\circ \text{ (see Figure 15.3).}$$

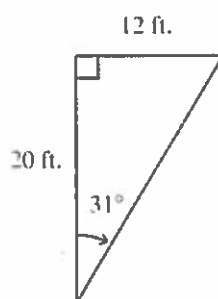


Figure 15.3

**Example**

A security camera is installed in a museum hallway. It is 10 feet away from the middle of a 30-foot section of wall that it must monitor (see Figure 15.4). How many degrees should it pan to the left and to the right?

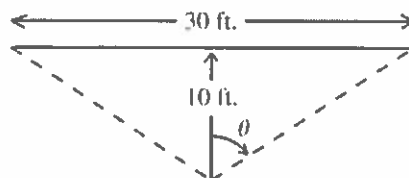


Figure 15.4

In this case, the side opposite the angle  $\theta$  is  $O = 15$  feet, so  $\theta = \tan^{-1}\left(\frac{15}{10}\right) \approx 56.3^\circ$ . Thus, the camera should turn  $56.3^\circ$  in each direction.

**Example**

A beam is to be cut to strengthen a wooden rectangle, as illustrated in Figure 15.5. At what angles  $\theta_1$  and  $\theta_2$  should the two ends of the beam be cut? Suppose that the space inside the rectangle is 12 feet by 9 feet 4 inches.

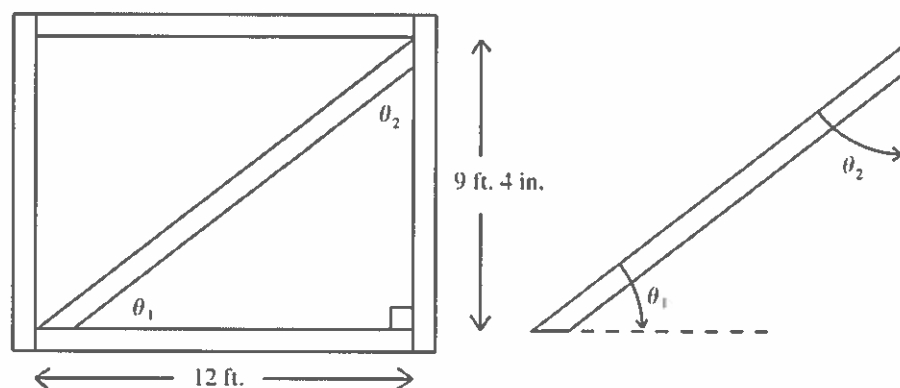


Figure 15.5

For angle  $\theta_1$ , the opposite length is  $O = 9'4'' = 9.333$  feet and the adjacent side is  $A = 12$  feet. Thus,  $\theta_1 = \tan^{-1}\left(\frac{9.333}{12}\right) \approx 37.87^\circ$ . Angle  $\theta_2$  is the complement,  $\theta_2 \approx 90 - 37.87 = 52.13^\circ$ . Of course, we could always calculate  $\theta_2 = \tan^{-1}\left(\frac{12}{9.333}\right) \approx 52.13^\circ$  for the same answer.

## ► Practice

1. A ship is 19 miles east and 52 miles south of its destination, as shown in Figure 15.6. What heading should the ship make to aim straight for its destination?

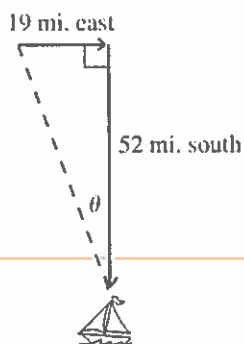


Figure 15.6

2. A swimming pool is 30 feet long. It is 3 feet deep at the shallow end and 12 feet deep at the deep end. If the bottom of the pool is flat, as shown in Figure 15.7, what is the angle of its slope?

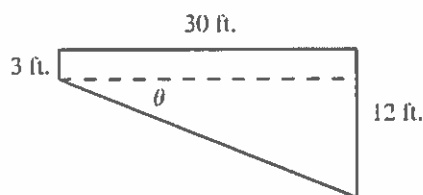


Figure 15.7

3. A searchlight must monitor an 800-foot prison wall. If the light is mounted 100 feet back from the middle of the wall, as shown in Figure 15.8, how many degrees of movement must it have?

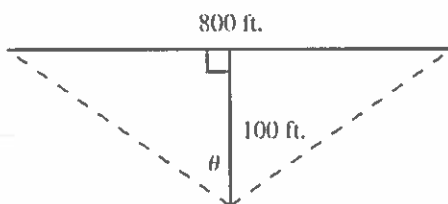


Figure 15.8

4. There are 15 feet of space to fit a ramp that must rise up the 2 feet 3 inches of a few steps. What would the angle of this ramp be?
5. On one side of a street is a 250-foot-tall building. Directly across the street, 100 feet away, a 320-foot-tall building will be constructed. How much will the view of the smaller building be obstructed. In other words, what will be the angle from the top of the first building to the top of the second?

## FINDING ANGLES

6. Suppose regulations prohibit a car's headlights from aiming further than 100 feet up the road. If the headlights are 3 feet off the ground, what is the least angle below the horizontal that they may be aimed?
7. A 3-mile-long road winds down a hill that is 1,500 feet tall. If the slope of the road were constant, what would be the angle? (Hint: Use 1 mile = 5,280 feet.)
8. A roof has a rise of 6 feet and a run of 15 feet. What is the angle of the roof?
9. A moving truck comes with a 14-foot-long ramp. If the loading space is 3 feet off the ground, what will the angle of the ramp be?
10. A spotlight is mounted 20 feet away from a mural. The light should shine at a spot 15 feet up the wall. At what angle should the spotlight be aimed?
11. Suppose a 40-foot tower is built to guard a gate that is 70 feet away. How many degrees below the horizontal is the direct path from the top of the tower to the gate?
12. A 25-foot-long guide wire will be attached to the top of a 24-foot-tall pole. What will be the angle between the wire and the pole?
13. How much will a 40-foot ladder lean if set 6 feet from the wall? That is, give the angle the ladder makes with the floor.
14. A college student would like to launch a potato from his dorm window to one that is three stories higher (30 feet) in a building 35 feet away. What is the angle of the straight line between the two windows?
15. Suppose a roof rises up a height of 15 feet as it runs horizontally 16 feet. What is the angle of the roof?
16. Good drainage is provided when the ground slopes at 1 inch down for every 10 feet over. What angle is this?
17. A diagonal board will be cut to fit into a rectangle that is 8 feet tall and 3.5 feet wide on the inside. At what angles should the ends of the board be cut?
18. A diagonal board will be cut to fit into a rectangle that is 4 feet 2 inches by 3 feet 4 inches. At what angles should the ends of the board be cut?
19. A ship is 4 miles off shore. Assuming that the shore is straight, what is the angle to a point 1.7 miles up the shore?

- 20.** A backyard loses 10 feet in elevation as it slopes down 22 feet (measured along the diagonal). What is the grade (the slope) of the ground?
- 21.** A video camera is set up 100 feet back from the center of a stage. Suppose the camera must be able to turn to point directly at any spot on the 40-foot-wide stage. What is the largest angle that it must be turned to either side?
- 22.** A 3-foot-wide door is attached to a doorway that is 1 foot 8 inches from a corner. How wide can the door be opened? That is, what is the measure of the obtuse angle when the door is opened widest?
- 23.** A woman stands in her yard. To the east is a stand of trees 30 feet tall and 25 feet away. To the west is her house, which is 40 feet tall and 35 feet away. How many degrees of sky can she see from where she stands?
- 24.** At what angle would you look up to see the top of a mountain if it has 2,000 feet more elevation than where you stand and is 2.3 miles away?
- 25.** A sloped driveway goes up 4 feet and is 22 feet long (measured along the diagonal ground). What is its grade?

# LESSON

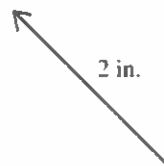
# 16



# Vectors

In this lesson, we explore the concept of a vector. We see how to break a vector down into component vectors—pieces that make for easy computation. We also explore a variety of word problems that are best solved by using vectors.

A **vector** is a length with a direction. Vectors are often depicted as arrows that point in the direction and have the correct length. The vector with a 2-inch length and northwest direction is shown in Figure 16.1.



**Figure 16.1**

In physics, vectors are used to represent forces. The arrow points in the direction of the push or pull, and the length represents the **magnitude** (strength) of the force. In navigation, vectors represent the components of journeys. The length represents the distance traveled in the direction of the arrow. Vectors can also represent the flow of wind and water, the stresses on objects, and anything else that has both a quantity and a direction.

There are several ways to represent a vector. One is to draw an arrow with the length and direction, as described above. Another is to give the length as a number and the direction as an angle. If we use the usual notation of  $0^\circ$  pointing straight to the right and positive angles measured counterclockwise, then the vector with length 40 and direction  $200^\circ$  would be as illustrated in Figure 16.2.

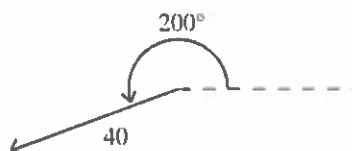


Figure 16.2

Usually, we consider that north is straight up (along the positive  $y$ -axis), south is straight down, east is to the right (along the positive  $x$ -axis), and west is to the left. Using these, the vector in Figure 16.2 is closest to pointing west. More specifically, it is  $20^\circ$  south of west. This is how the directions of navigation vectors are usually described (see Figure 16.3).

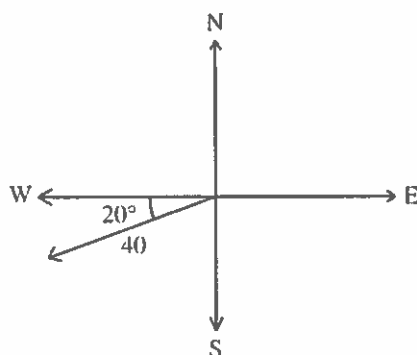


Figure 16.3

Unfortunately for us, navigators consider north to be  $0^\circ$  and then measure their angles clockwise, so northeast is  $45^\circ$ , east is  $90^\circ$ , south is  $180^\circ$ , and west is  $270^\circ$ . Because this conflicts with the way mathematicians measure angles (measured counterclockwise with east  $0^\circ$ ), we shall stick with the “ $20^\circ$  south of west” notation.

A last way to represent a vector is to put its start at the origin and give the coordinates of its endpoint. This is where trigonometry is used. The vector in Figure 16.3 can be used as the hypotenuse of a right triangle, as shown in Figure 16.4. The angle to the nearest axis, here  $20^\circ$ , is called the **reference angle**.

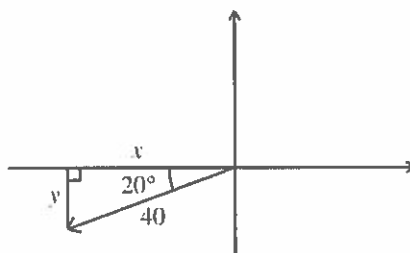


Figure 16.4



$$\sin(20^\circ) = \frac{y}{40}, \text{ so } y = 40 \cdot \sin(20^\circ) \approx 13.68$$

$$\cos(20^\circ) = \frac{x}{40}, \text{ so } x = 40 \cdot \cos(20^\circ) \approx 37.59$$

Thus, this vector can be described by the endpoint  $(-37.59, -13.68)$  or as a combination of the two *component vectors* shown in Figure 16.5. The vector can be broken into a component vector of 37.59 west, and another component of 13.68 south.

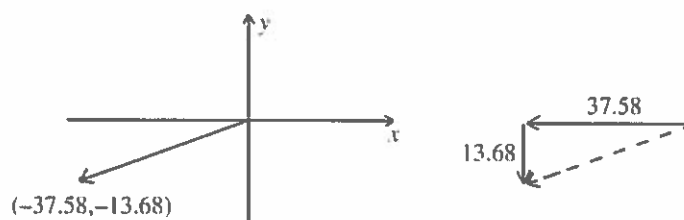


Figure 16.5

### Example

What is the length and direction of the vector that runs from the origin to the point  $(8, -15)$ ?

We sketch the vector in Figure 16.6.

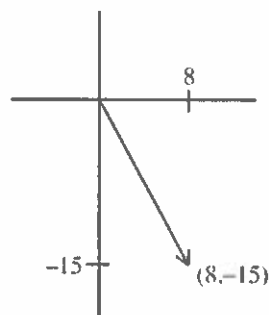


Figure 16.6

The length  $L$  of this vector can be found by the Pythagorean theorem.

$$L^2 = 8^2 + 15^2, \text{ so } L = \sqrt{64 + 225} = \sqrt{289} = 17$$

The reference angle  $\theta$  is formed by the vector and the nearest axis, which in this case is the negative  $y$ -axis. The length opposite this angle is 8 and the adjacent length is 15, so

$$\tan(\theta) = \frac{8}{15}, \text{ thus, } \theta = \tan^{-1}\left(\frac{8}{15}\right) \approx 28^\circ$$

This vector has a length of 17 and points  $28^\circ$  east of south. We could also say that the angle of this vector is  $298^\circ$ . This is depicted in Figure 16.7.

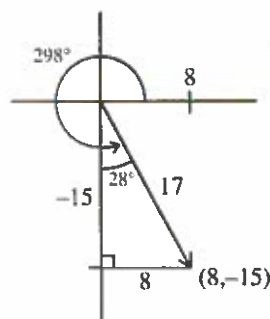


Figure 16.7

In general, the length of the vector from the origin to  $(x,y)$  is  $L = \sqrt{x^2 + y^2}$ . The reference angle can be found with an inverse tangent.

### Example

Find the endpoint of the vector that starts at the origin, has a length of 16, and makes an angle of  $143^\circ$  with the positive  $x$ -axis (measured counterclockwise).

If our vector had a length of 1, then the endpoint would be on the unit circle. The  $y$ -coordinate would be  $\sin(143^\circ) \approx 0.602$  and the  $x$ -coordinate would be  $\cos(143^\circ) \approx -0.799$ . Because of similar triangles, all we need to find our endpoint is to multiply each of these coordinates by 16. Because  $16(-0.799) = -12.784$  and  $16(0.602) = 9.632$ , the endpoint of this vector is  $(-12.784, 9.632)$ . This is illustrated in Figure 16.8.

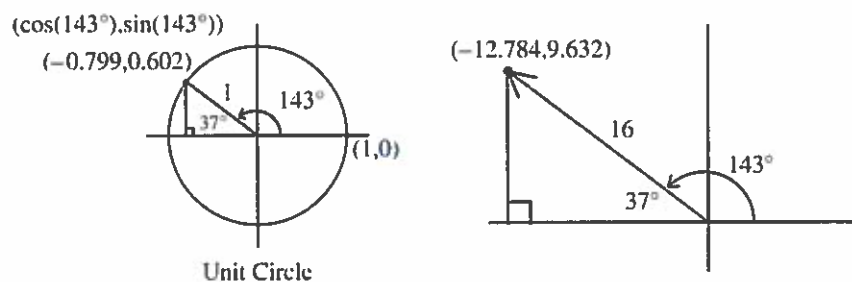


Figure 16.8

In general, the endpoint of a vector with angle  $\theta$  and length  $L$  is  $(L \cdot \cos(\theta), L \cdot \sin(\theta))$ .

### ► Practice

Find the endpoint of each vector if it starts at the origin.

1. length 70, angle  $79^\circ$
2. length 30, angle  $308^\circ$

3. length 68.4, angle  $195.2^\circ$
4. length 400,  $10^\circ$  north of west
5. length 141,  $21^\circ$  west of south

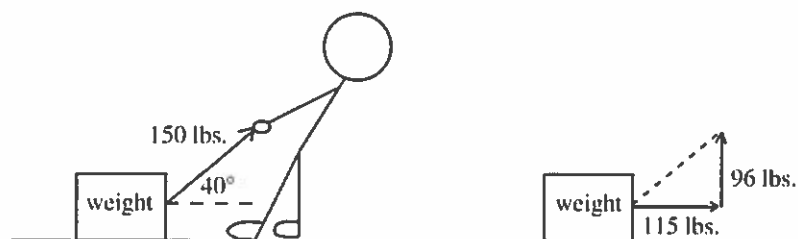
Find the length and direction of the vector from the origin to the given point.

6.  $(-12, -3)$
7.  $(2, 7)$
8.  $(23, -9)$
9.  $(-208, 119)$
10.  $(1.926, -3.17)$

### ► Component Vectors

The components of a vector can be very useful in solving problems. When an object is being pushed or pulled, only the component that points in the direction of movement actually contributes to the movement.

For example, suppose a heavy weight sits on the ground. A person ties a rope to the weight and pulls so that the rope makes an angle of  $40^\circ$  with the ground. Suppose that the person pulls with 150 pounds of force, enough force to lift a 150-pound object (see Figure 16.9).



**Figure 16.9**

The vector with magnitude 150 pounds and  $40^\circ$  can be broken into two component vectors. The one in the  $x$ -direction is  $150 \cdot \cos(40^\circ) \approx 115$  pounds. The one in the  $y$ -direction is  $150 \cdot \sin(40^\circ) \approx 96$  pounds. This means that only 115 pounds of force is pulling the weight toward the person. The rest of the force is acting to lift the weight off the ground. Whether or not this is enough to make the weight move depends on how heavy it is and how much friction the floor has.

If the person were to pull with the same 150 pounds of force, but at a  $30^\circ$  angle instead, then the  $x$ -component of the vector would be  $150 \cdot \cos(30^\circ) \approx 130$  pounds of force. This is illustrated in Figure 16.10. This explains why people bend down low to push cars and pull heavy objects.

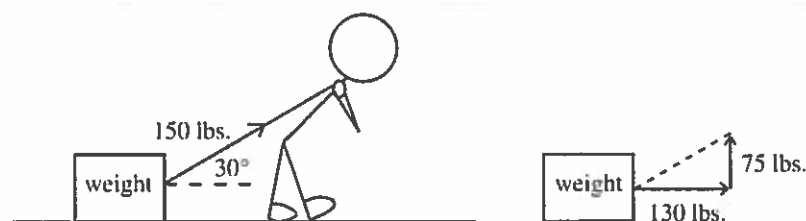


Figure 16.10

Similarly, if an object is on a slope, only a component of its weight acts to pull it down the slope.

### Example

Suppose a 200-pound weight is put on a  $15^\circ$  ramp, as illustrated in Figure 16.11. How much force is pulling the weight down the slope?

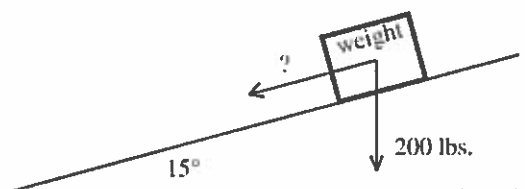


Figure 16.11

Gravity is pulling the weight straight down. The direction down the slope is  $15^\circ$  from horizontal, thus  $75^\circ$  from straight down, as illustrated in Figure 16.12.

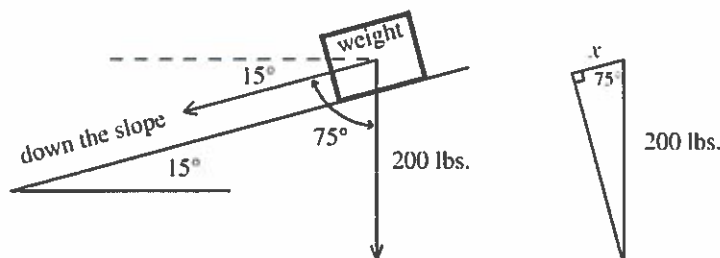


Figure 16.12

We want to know the length  $x$  of the component vector that points down the slope. This is adjacent to the  $75^\circ$  angle of a right triangle with hypotenuse 200. Thus,

$$\cos(75^\circ) = \frac{x}{200} \text{ and so } x = 200 \cdot \cos(75^\circ) \approx 51.8$$

Thus, about 51.8 pounds of the object's weight are pushing in the direction of the slope. If a person pushed up the slope with a force of more than 51.8 pounds, then the weight would slide uphill (after friction was overcome).

### ► Practice

Suppose the person in Figure 16.13 is trying to pull the heavy object with  $F$  pounds of force at an angle of  $\theta$  from the horizontal. Find the component of the force that is going toward sliding the object across the floor.

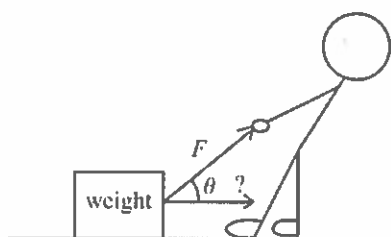


Figure 16.13

11.  $\theta = 20^\circ$ ,  $F = 150$  pounds
12.  $\theta = 5^\circ$ ,  $F = 80$  pounds
13.  $\theta = 60^\circ$ ,  $F = 200$  pounds
14.  $\theta = 37^\circ$ ,  $F = 180$  pounds
15.  $\theta = 8.6^\circ$ ,  $F = 130$  pounds

Suppose a weight of  $W$  pounds is set on a ramp with angle  $\theta$ , as illustrated in Figure 16.14. How much of the weight is going down the slope?

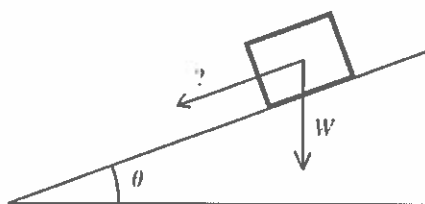


Figure 16.14

16.  $W = 100$  pounds,  $\theta = 15^\circ$
17.  $W = 200$  pounds,  $\theta = 18^\circ$

18.  $W = 40$  pounds,  $\theta = 42^\circ$

19.  $W = 1,000$  pounds,  $\theta = 5^\circ$

20.  $W = 100$  pounds,  $\theta = 87^\circ$

## ► Adding Vectors

To add two vectors, convert them into components and add each component separately. For example, the sum of  $(8,3)$  and  $(-4,4)$  is  $(8 + (-4), 3 + 4) = (4,7)$ . If the two vectors represent two trips, the sum will be the result of traveling along first one path and then the second. This is illustrated in Figure 16.15.

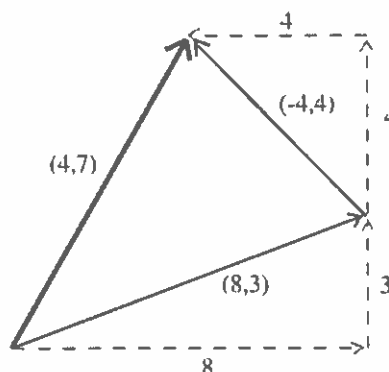


Figure 16.15

If the two vectors represent forces, the sum represents the overall effect of applying both forces at the same time to a single object. This is illustrated in Figure 16.16.

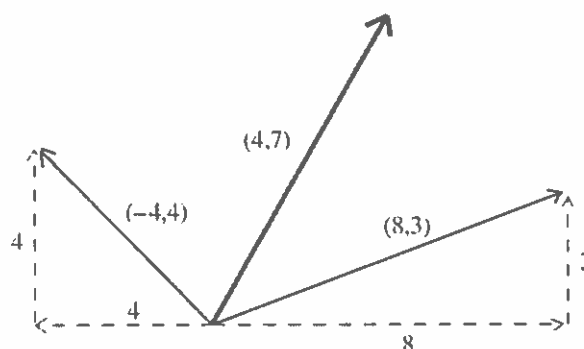


Figure 16.16

### Example

Suppose a hiker travels  $15^\circ$  west of north for 5 miles, and then  $10^\circ$  north of west for 8 miles. How far and in what direction is the hiker from the starting point?

The first vector has magnitude 5 and direction  $105^\circ$ , so the components are  $(5 \cdot \cos(105^\circ), 5 \cdot \sin(105^\circ)) = (-1.3, 4.8)$ . The second vector has magnitude 8 and direction  $170^\circ$ , so its components are  $(8 \cdot \cos(170^\circ), 8 \cdot \sin(170^\circ)) = (-7.9, 1.4)$ . The sum is thus  $(-1.3 - 7.9, 4.8 + 1.4) = (-9.2, 6.2)$ . The magnitude of this vector is  $\sqrt{(-9.2)^2 + (6.2)^2} = \sqrt{123.08} \approx 11.1$ , so the hiker is now 11.1 miles away from the starting point. The angle  $\theta$  this vector makes with the  $x$ -axis is  $\theta = \tan^{-1}\left(\frac{6.2}{-9.2}\right) \approx 34^\circ$ . Looking at the sketch in Figure 16.17, this direction is  $34^\circ$  north of west.

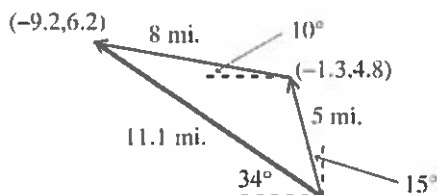


Figure 16.17

### Example

Two people pull on a sack of potatoes. The angle between them is  $40^\circ$ . One person pulls with 100 pounds of force and the other with 70 pounds. In which direction will the sack move and with how much force?

To make things easier, we can suppose that the larger force is headed in the  $0^\circ$  direction. We thus have the situation in Figure 16.18.

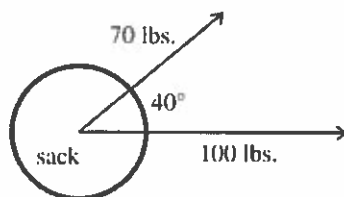


Figure 16.18

The component form for the larger vector is  $(100, 0)$ . The component form of the second vector is  $(70 \cdot \cos(40^\circ), 70 \cdot \sin(40^\circ)) = (53.6, 45)$ . The sum is thus  $(153.6, 45)$ . This has a magnitude of  $\sqrt{(153.6)^2 + 45^2} = \sqrt{25,617.96} \approx 160$  pounds. The angle of this vector is  $\theta = \tan^{-1}\left(\frac{45}{153.6}\right) \approx 16.3^\circ$ .

Thus, the sack will move in a direction between the two people that is  $16.3^\circ$  from the person who is pulling with 100 pounds of force. It will move as if pulled in that direction by 160 pounds of weight.

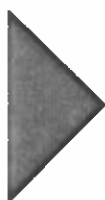
**► Practice**

- 21.** Suppose a person walks 30 meters north and then 44 meters northwest. What is the final distance and direction from the starting point?
- 22.** A boat travels  $14^\circ$  north of east for 16 miles, and then  $19^\circ$  north of east for 20 miles. How far is the boat from where it started?
- 23.** A hiker travels  $25^\circ$  south of west for 4 miles and then  $10^\circ$  north of west for 3 miles. In which direction should she head if she wants to go directly back to where she started?
- 24.** A car is being driven through the desert. It goes for 12 miles in the direction  $5^\circ$  south of east. Then the car is turned and driven for 20 miles heading  $7^\circ$  north of east. Finally, it is driven for 9 miles heading  $30^\circ$  north of east. How far is it from where it started? In what direction is it from where it started?
- 25.** Suppose two people are pulling on a heavy statue. One of them pulls with 180 pounds of force and the other with 120 pounds of force. The angle between the two people is  $25^\circ$ . With how much force is the statue being pulled in the direction of its movement? What is the direction of its movement?



# LESSON

# 17



# The Law of Sines

In this lesson, we prove a powerful theorem called the Law of Sines. This enables us to find the angles and sides of non-right triangles. With the trigonometric functions, we can measure the sides and angles of right triangles. To study a non-right triangle, we break it into right triangles. The first major result in this area is called the Law of Sines.

To begin, take any triangle and suppose the lengths of its sides are  $A$ ,  $B$ , and  $C$ . Suppose the measure of the angle opposite side  $A$  is  $a$ , the angle opposite  $B$  is  $b$ , and the angle opposite  $C$  is  $c$ . This is illustrated in Figure 17.1.

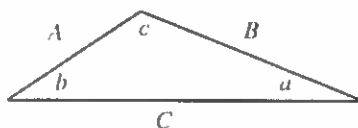


Figure 17.1

If the proof on the next two pages is difficult to follow, you may skip straight to the Law of Sines on page 164.

We can draw a line from the vertex at angle  $c$  that makes a right angle with side  $C$ . The length of this line is called the height  $h$  of the triangle (see Figure 17.2).

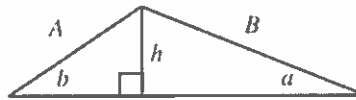


Figure 17.2

This new line creates two right triangles, shown separately in Figure 17.3.

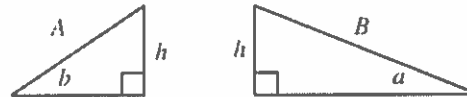


Figure 17.3

The one on the left has an angle of measure  $b$ . Opposite this angle is a side of length  $h$ . The hypotenuse has length  $A$ . Thus,

$$\sin(b) = \frac{h}{A} \text{ so } h = A \cdot \sin(b)$$

The other right triangle has angle  $a$ . Here,

$$\sin(a) = \frac{h}{B} \text{ so } h = B \cdot \sin(a)$$

If we put the two of these equations together, we get

$$B \cdot \sin(a) = h = A \cdot \sin(b)$$

If we divide both sides by  $AB$ , we get part of the Law of Sines:

$$\frac{\sin(a)}{A} = \frac{\sin(b)}{B}$$

The full Law of Sines is

$$\frac{\sin(a)}{A} = \frac{\sin(b)}{B} = \frac{\sin(c)}{C}$$

proving that  $\frac{\sin(b)}{B} = \frac{\sin(c)}{C}$  is done by the same method as above. It can be a little bit tricky, however, if one of the angles is obtuse (greater than  $90^\circ$ ), as is the case with angle  $c$  in our example. First, we rotate the triangle so that side  $A$  is across the bottom. When we try to make a right angle from the vertex of angle  $a$  to the side  $A$ , we must extend side  $A$  and have the right angle outside of the triangle, as is illustrated in Figure 17.4.

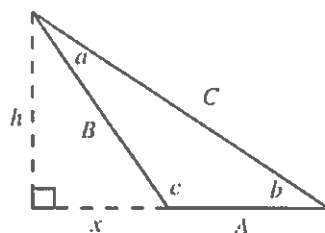


Figure 17.4

The two right triangles that are formed have angles of measure  $b$  and  $180^\circ - c$ , as shown separately in Figure 17.5.

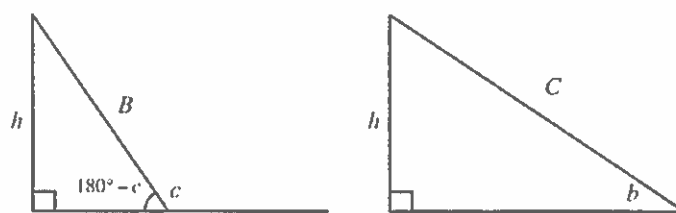


Figure 17.5

Thus, we get

$$\sin(b) = \frac{h}{C}, \text{ so } h = C \cdot \sin(b) \text{ and}$$

$$\sin(180^\circ - c) = \frac{h}{B}, \text{ so } h = B \cdot \sin(180^\circ - c)$$

We will see in just a moment that  $\sin(180^\circ - c) = \sin(c)$ ; thus,  $C \cdot \sin(b) = h = B \cdot \sin(c)$ , so  $\frac{\sin(b)}{B} = \frac{\sin(c)}{C}$ , as desired. The proof comes from looking at the unit circle in Figure 17.6.

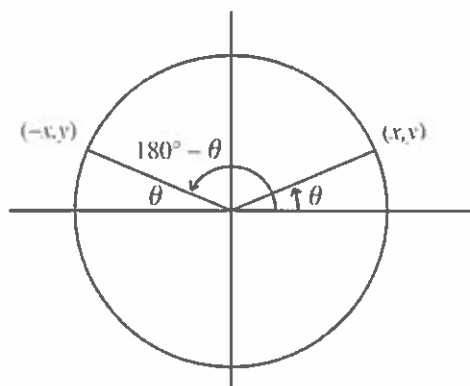


Figure 17.6

## THE LAW OF SINES

The point on the unit circle that corresponds to the angle  $\theta$  has the same height (sine) as the point that corresponds to the point  $180^\circ - \theta$ . This is why  $\sin(180^\circ - \theta) = \sin(\theta)$ . Thus, we have completely proven the Law of Sines:

$$\frac{\sin(a)}{A} = \frac{\sin(b)}{B} = \frac{\sin(c)}{C}$$

### Tip

Sometimes the Law of Sines is given instead as the equivalent or, equivalently:

$$\frac{A}{\sin(a)} = \frac{B}{\sin(b)} = \frac{C}{\sin(c)}$$

This can be used to find the length of a side of a triangle, provided that we know the measure of one side and two angles.

### Example

Find the length  $x$  of the side in Figure 17.7.

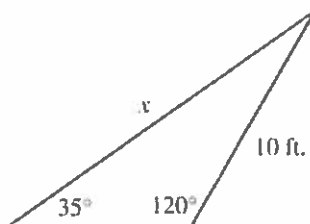


Figure 17.7

The length  $x$  is opposite the  $120^\circ$  angle, and the 10-foot side is opposite the  $35^\circ$  angle. Thus, we can use the law of sines:

$$\frac{\sin(a)}{A} = \frac{\sin(b)}{B}$$

$$\frac{\sin(35^\circ)}{10} = \frac{\sin(120^\circ)}{x}$$

The only important thing is that the angle and side in each fraction are opposite each other. With cross multiplication, we get

$$x \cdot \sin(35^\circ) = 10 \cdot \sin(120^\circ); \text{ thus, } x = \frac{10 \cdot \sin(120^\circ)}{\sin(35^\circ)} \approx 15.1$$

Thus, the length of this side is about 15.1 feet long.

**Example**

Find the length  $x$  in Figure 17.8.

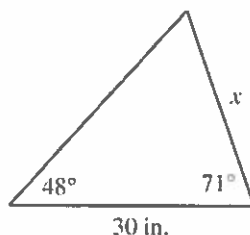


Figure 17.8

We can't immediately use the Law of Sines because we don't know the measure of the angle opposite the 30-inch side. However, this can be easily calculated because the angles of a triangle must add up to  $180^\circ$ . The angle opposite the 30-inch side measures  $180^\circ - 48^\circ - 71^\circ \approx 61^\circ$ . With this, we can now use the Law of Sines:

$$\frac{\sin(61^\circ)}{30} = \frac{\sin(48^\circ)}{x}$$

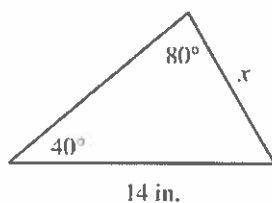
$$x \cdot \sin(61^\circ) = 30 \cdot \sin(48^\circ)$$

$$x = \frac{30 \cdot \sin(48^\circ)}{\sin(61^\circ)} \approx 25.5 \text{ inches}$$

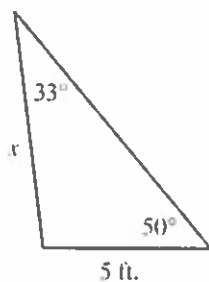
**► Practice**

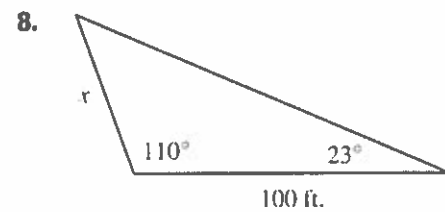
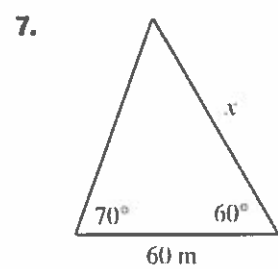
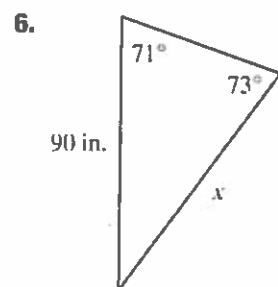
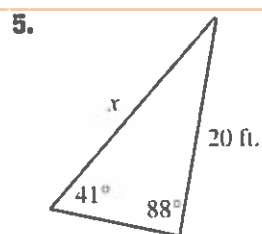
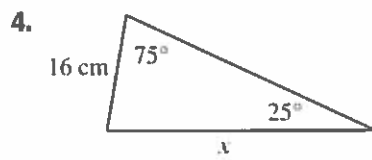
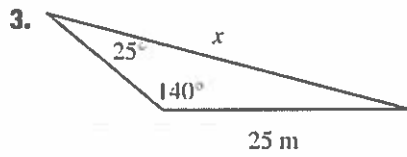
Find the length  $x$  in each of the following triangles.

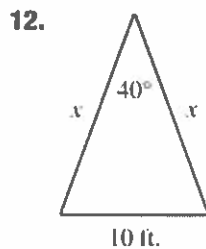
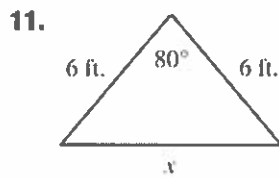
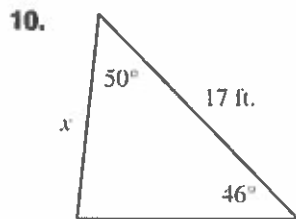
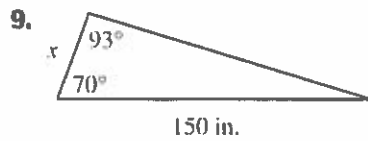
1.



2.







### ► Finding Angles with the Law of Sines

We can use the Law of Sines to find any side of a triangle if we know two angles and one side. If we try to find the measure of an angle given two sides and one angle, however, a problem arises. For example, suppose we have the triangle in Figure 17.21. What is the measure of angle  $x$ ?

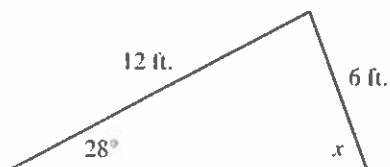


Figure 17.21

Because we have two pairs of opposite angles and sides, we can set up the Law of Sines:

$$\frac{\sin(x)}{12} = \frac{\sin(28^\circ)}{6}$$

$$\sin(x) = \frac{12 \cdot \sin(28^\circ)}{6} = 2 \cdot \sin(28^\circ) \approx 0.9389$$

Our instinct is to evaluate  $x$  with the inverse sine function:

$$x \approx \sin^{-1}(0.9389) \approx 70^\circ$$

If we know that Figure 17.21 is drawn accurately, then this is the measure of angle  $x$ . However, if Figure 17.21 is not drawn accurately, then it is possible that the triangle is really the one in Figure 17.22.

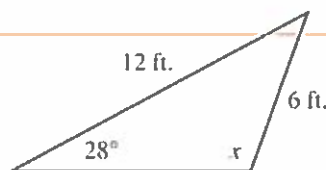


Figure 17.22

In this case,  $x$  is an obtuse angle (greater than  $90^\circ$ ) and thus could not be  $70^\circ$ .

In Figure 17.22, the measure of angle  $x$  is  $110^\circ$ , the supplement of  $70^\circ$ . This is because  $\sin(110^\circ) = \sin(70^\circ)$  as discussed after Figure 17.6:

$$\sin(180^\circ - \theta) = \sin(\theta)$$

The inverse sine function will output angles only between  $-90^\circ$  and  $90^\circ$ . That is to say, it only outputs acute angles. If we are seeking the measure of an acute angle  $\theta$  with  $\sin(\theta) = r$  for a specific positive ratio  $r$ , then  $\theta = \sin^{-1}(r)$ . If we want the measure of an obtuse angle  $\theta$  with  $\sin(\theta) = r$ , then  $\theta = 180^\circ - \sin^{-1}(r)$ .

### Example

Find the measure of the obtuse angle  $x$  in Figure 17.23.

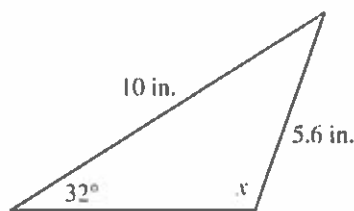


Figure 17.23

We have  $x$  opposite the side of length 10 inches and  $32^\circ$  opposite the side of length 5.6 inches. Thus, we can use the Law of Sines:

$$\frac{\sin(x)}{10} = \frac{\sin(32^\circ)}{5.6}$$



so,

$$\sin(x) = \frac{10}{5.6} \cdot \sin(32^\circ) \approx 0.9463$$

Using the inverse sine function, we get

$$\sin^{-1}(0.9463) \approx 71.1^\circ$$

This cannot be the measure of angle  $x$  because  $x$  is obtuse. Thus,  $x$  must be the supplement of this angle.

$$x \approx 180^\circ - 71.1^\circ = 108.9^\circ$$

### Example

Suppose the triangle in Figure 17.24 is drawn accurately. Find the measure of angle  $x$ .

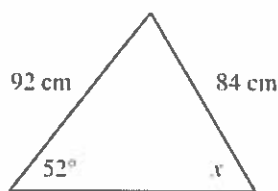


Figure 17.24

Here, we use the Law of Sines:

$$\frac{\sin(x)}{92} = \frac{\sin(52^\circ)}{84}$$

$$\sin(x) = \frac{92}{84} \cdot \sin(52^\circ) \approx 0.8631$$

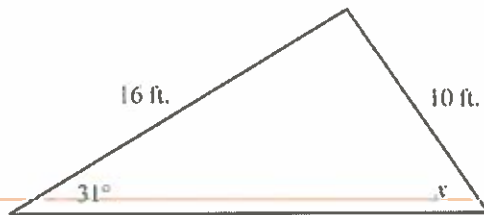
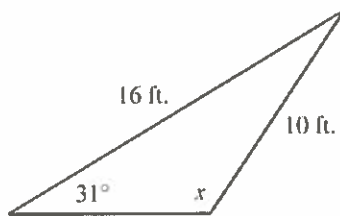
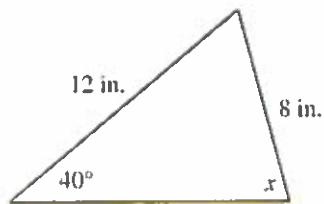
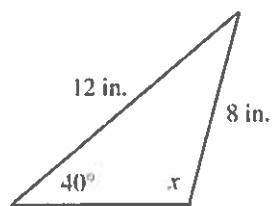
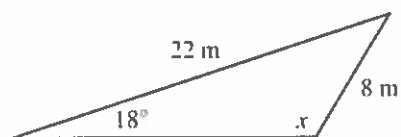
Now,

$$\sin^{-1}(0.8631) \approx 59.7^\circ$$

Because  $x$  is an acute angle, this measure is correct:  $x \approx 59.7^\circ$ . If  $x$  had been obtuse, then its measure would be  $180^\circ - 59.7^\circ = 120.3^\circ$ .

**► Practice**

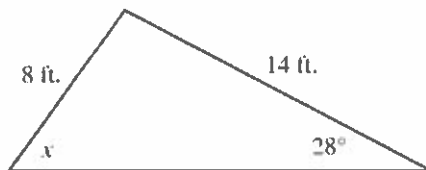
Find the measure of angle  $x$  in each triangle. You may assume that each triangle is drawn accurately. In other words,  $x$  is acute or obtuse if it appears so.

**13.****14.****15.****16.****17.**

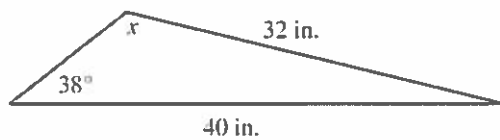
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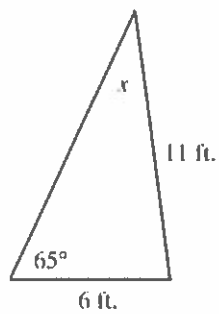
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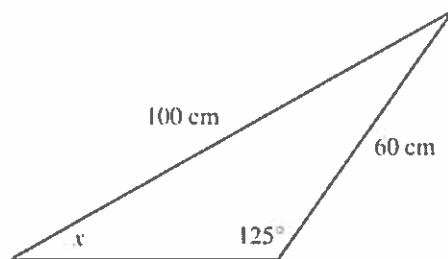
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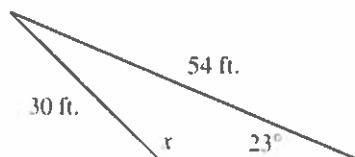
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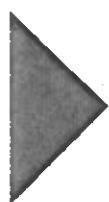
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# LESSON

# 18



# The Law of Cosines

**W**hen we know the angles of a triangle and one of the sides, we can figure out the lengths of the other sides with the Law of Sines. In this lesson, we see how a new theorem, the **Law of Cosines**, enables us to find the third length of a triangle when we know the lengths of the other two sides and the angle between them. For example, suppose we know the lengths  $A$  and  $B$  of the triangle in Figure 18.1 and the measure  $\theta$  of the angle between them. We would like to find the length  $C$  of the third side.

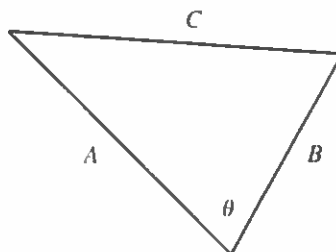


Figure 18.1

## Law of Cosines

$$C^2 = A^2 + B^2 - 2AB \cdot \cos\theta$$

We can prove this with vectors (if the proof is confusing, feel free to skip to the examples on page 175). To make things easier, rotate the triangle so that the angle  $\theta$  is in standard position, as shown in Figure 18.2.

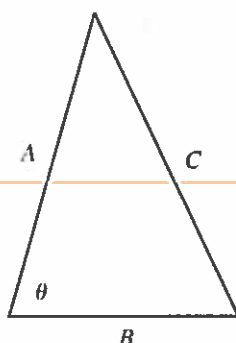


Figure 18.2

Let the origin be the right endpoint of the side with length  $B$ . If we look at the side of length  $B$  as a vector, then its components are  $(-B, 0)$ , because it goes a distance of  $B$  to the left and does not go up or down at all. The side with length  $A$  goes up at an angle of  $\theta$ , so as a vector, its components are  $(A \cdot \cos(\theta), A \cdot \sin(\theta))$ . The sum of these two vectors is thus  $(A \cdot \cos(\theta) - B, A \cdot \sin(\theta))$ , as illustrated in Figure 18.3.

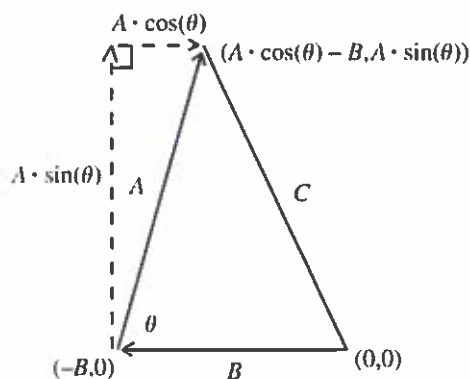


Figure 18.3

The side with length  $C$  goes from the origin  $(0, 0)$  to the point  $(A \cdot \cos(\theta) - B, A \cdot \sin(\theta))$ . This means that it is the hypotenuse of a triangle with legs of length  $|A \cdot \cos(\theta) - B|$  and  $A \cdot \sin(\theta)$ , as shown in Figure 18.4. Because we don't know if  $A \cdot \cos(\theta)$  or  $B$  is bigger, we use absolute values to describe the length of the bottom of this triangle. When we square this length, it won't matter whether  $A \cdot \cos(\theta) - B$  was positive or negative.

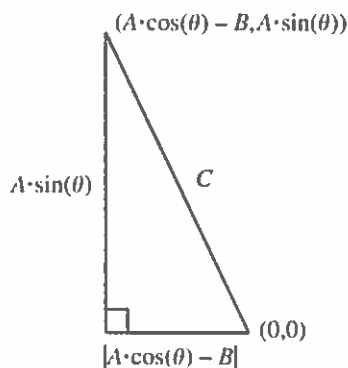


Figure 18.4

We can now find the length  $C$  with the Pythagorean theorem:

$$C^2 = (A \cdot \sin(\theta))^2 + (A \cdot \cos(\theta) - B)^2$$

$$C^2 = A^2 \cdot \sin^2(\theta) + A^2 \cdot \cos^2(\theta) - 2AB \cdot \cos(\theta) + B^2$$

We can factor the  $A^2$  out of the first two terms on the right.

$$C^2 = A^2(\sin^2(\theta) + \cos^2(\theta)) - 2AB \cdot \cos(\theta) + B^2$$

If we use  $\sin^2(\theta) + \cos^2(\theta) = 1$ , proved in Lesson 6, we end up with the Law of Cosines:

$$C^2 = A^2 - 2AB \cdot \cos(\theta) + B^2$$

Usually, this is written as  $C^2 = A^2 + B^2 - 2AB \cdot \cos(\theta)$ . We can use this to quickly find the lengths of triangles.

### Example

Find the length  $x$  in Figure 18.5.

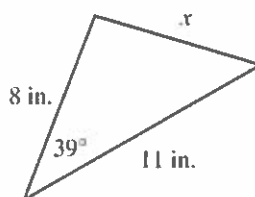


Figure 18.5

We can use the Law of Cosines with  $A = 8$ ,  $B = 11$ ,  $C = x$ , and  $\theta = 39^\circ$ :

$$C^2 = A^2 + B^2 - 2AB \cdot \cos(\theta)$$

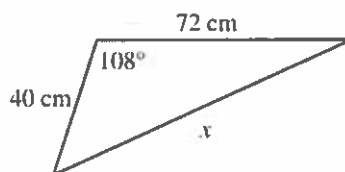
$$x^2 = 8^2 + 11^2 - 2(8)(11) \cdot \cos(39^\circ) \approx 48.22$$

$$x \approx \sqrt{48.22} \approx 6.9$$

The length is  $x \approx 6.9$  inches.

**Example**

Find the length  $x$  in Figure 18.6.

**Figure 18.6**

We use the Law of Cosines with  $A = 40$ ,  $B = 72$ ,  $C = x$ , and  $\theta = 108^\circ$ .

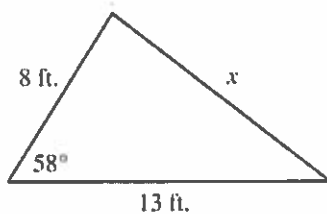
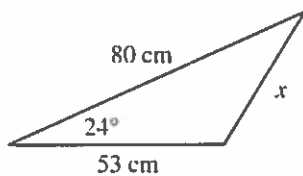
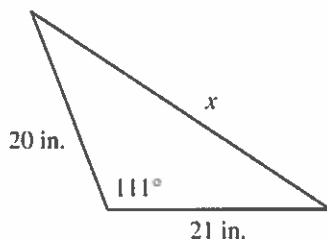
$$C^2 = A^2 + B^2 - 2AB \cdot \cos(\theta)$$

$$x^2 = 40^2 + 72^2 - 2(40)(72) \cdot \cos(108^\circ) \approx 8,563.94$$

$$x \approx \sqrt{8,563.94} \approx 92.5 \text{ cm}$$

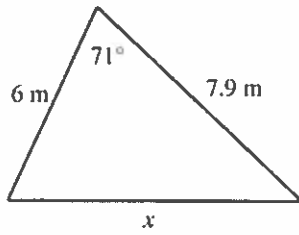
**► Practice**

Use the Law of Cosines to find the length  $x$  in each triangle.

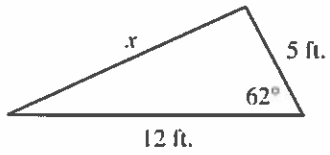
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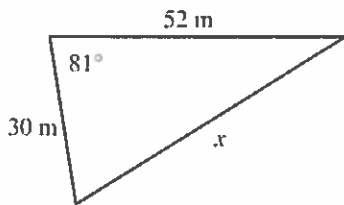
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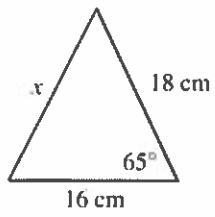
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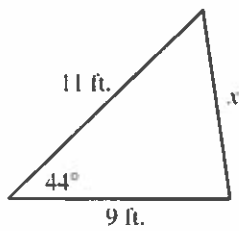
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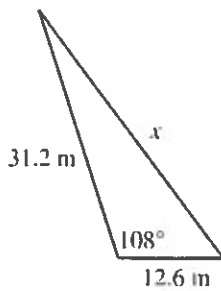
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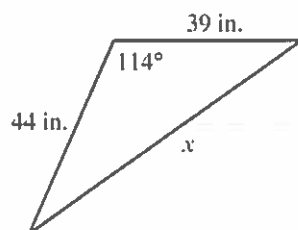
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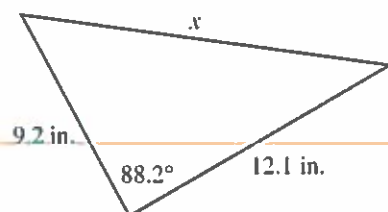
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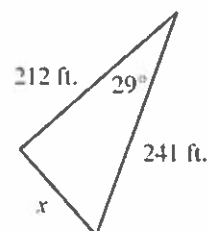
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11.



12.



### ► Finding Angles with the Law of Cosines

When we know the lengths of all the sides of a triangle, the Law of Cosines can find the measure of any angle. For example, let us find the measures of angles  $x$ ,  $y$ , and  $z$  in the triangle pictured in Figure 18.19.

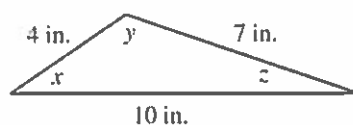


Figure 18.19

To find the measure of angle  $x$ , we use the Law of Cosines with  $A$  and  $B$  the adjacent edges and  $C$  the opposite edge. That is,  $A = 4$ ,  $B = 10$ ,  $C = 7$ , and  $\theta = x$ . Thus,

$$C^2 = A^2 + B^2 - 2AB \cdot \cos(\theta)$$

$$7^2 = 4^2 + 10^2 - 2(4)(10) \cdot \cos(x)$$

$$49 - 116 = -80 \cdot \cos(x)$$

$$\cos(x) = \frac{-67}{-80} = 0.8375$$

$$x = \cos^{-1}(0.8375) \approx 33.1^\circ$$

To find the measure of angle  $y$ , we use  $A = 4$ ,  $B = 7$ ,  $C = 10$ , and  $\theta = y$ . The only really important thing is to make  $C$  be the side opposite the angle we are trying to measure.

$$C^2 = A^2 + B^2 - 2AB \cdot \cos(\theta)$$

$$10^2 = 4^2 + 7^2 - 2(4)(7) \cdot \cos(y)$$

$$100 = 65 - 56 \cdot \cos(y)$$

$$\cos(y) = \frac{35}{-56} = -0.625$$

$$y = \cos^{-1}(-0.625) \approx 128.7^\circ$$

The numbers that come out of the inverse cosine function range from  $0^\circ$  to  $180^\circ$ . This means that we do not have to deal with acute and obtuse angles separately, as we will do in Lesson 19. The acute angle  $x$  is estimated at  $33.1^\circ$  and the obtuse angle  $y$  is estimated at  $128.7^\circ$  by the same process.

We could use the Law of Cosines to find the measure of angle  $z$ , but it would be much easier to use the fact that the three angles of a triangle add up to  $180^\circ$ . Thus,

$$z \approx 180^\circ - 33.1^\circ - 128.7^\circ = 18.2^\circ$$

### Example

Find the measure of the angle  $x$  illustrated in Figure 18.20.

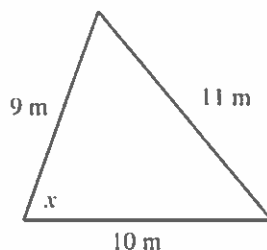


Figure 18.20

We use the Law of Cosines to find  $\theta = x$  with adjacent edges  $A = 9$  and  $B = 10$ , and opposite edge  $C = 11$ .

$$C^2 = A^2 + B^2 - 2AB \cdot \cos(\theta)$$

$$11^2 = 9^2 + 10^2 - 2(9)(10) \cdot \cos(x)$$

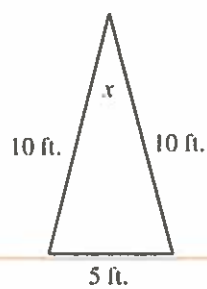
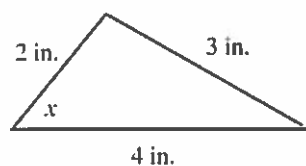
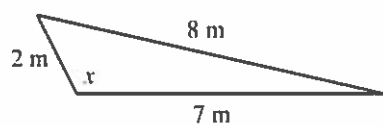
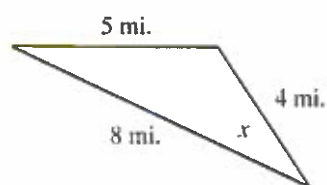
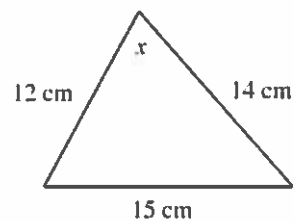
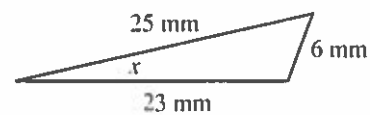
$$121 = 181 - 180 \cdot \cos(x)$$

$$\cos(x) = \frac{-60}{-180} = \frac{1}{3}$$

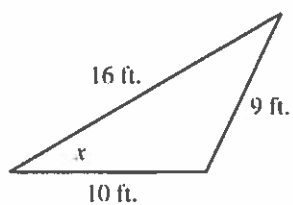
$$x = \cos^{-1}\left(\frac{1}{3}\right) \approx 70.5^\circ$$

**► Practice**

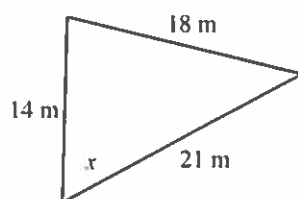
Use the Law of Cosines to find the measure of angle  $x$  in each triangle.

**13.****14.****15.****16.****17.****18.**

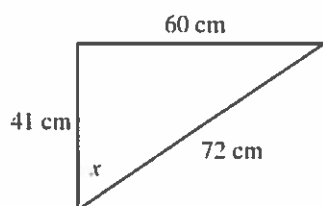
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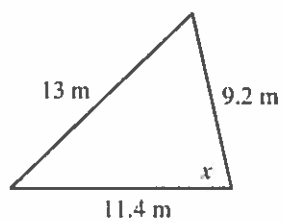
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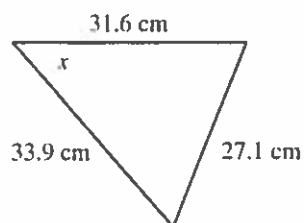
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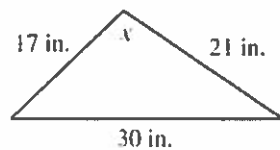
22.



23.



24.



### ► An Ambiguous Situation

If we know the lengths of two sides of a triangle and the angle between them, then we can use the Law of Cosines to find the length of the third side. If the angle we know is not between the two sides, then there will be ambiguity. For example, look at the two triangles in Figure 18.33.

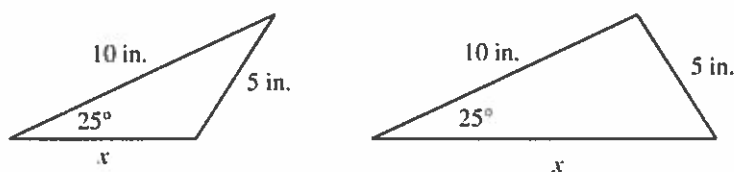


Figure 18.33

This situation was discussed at the end of the last lesson. The Law of Sines can be used to find the angles in the lower-right corners of the triangles. However, this process will result in two different angle measures: one acute and one obtuse. Similarly, the Law of Cosines can find the measure the sides labeled  $x$ , but it will result in two different numbers. Incidentally, this ambiguity relates to the fact that there is no Angle-Side-Side theorem in geometry, even though there is a Side-Angle-Side theorem.

If we use  $A = x$ ,  $B = 10$ ,  $C = 5$ , and  $\theta = 25^\circ$  (remember that the side labeled  $C$  must be opposite the given angle), then the Law of Cosines states:

$$C^2 = A^2 + B^2 - 2AB \cdot \cos(\theta)$$

$$5^2 = x^2 + 10^2 - 2x(10) \cdot \cos(25)$$

$$25 = x^2 + 100 - (18.13)x$$

$$x^2 - (18.13)x + 75 = 0$$

This is a quadratic equation that is not easily factored. Fortunately, it can be evaluated with the quadratic formula.

### Tip

Any quadratic equation  $ax^2 + bx + c = 0$  can be solved by the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

To solve  $x^2 - (18.13)x + 75 = 0$ , we use the quadratic formula with  $a = 1$ ,  $b = -18.13$ , and  $c = 75$ . Thus,

$$x = \frac{18.13 + \sqrt{(18.13)^2 - 4(75)}}{2} \approx 11.74 \text{ inches}$$

or

$$x = \frac{18.13 - \sqrt{(18.13)^2 - 4(75)}}{2} \approx 6.39 \text{ inches}$$

The first answer is the  $x$  for the triangle on the right of Figure 18.33, and the second is for the triangle on the left.

### Example

The triangle in Figure 18.34 is not drawn accurately. Find the two possible lengths for the side labeled  $x$ .

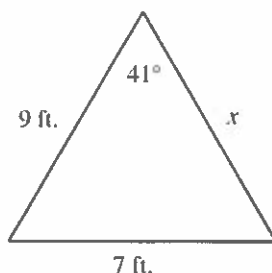


Figure 18.34

We use the Law of Cosines with  $A = x$ ,  $B = 9$ ,  $C = 7$ , and  $\theta = 41^\circ$ .

$$C^2 = A^2 + B^2 - 2AB \cdot \cos(\theta)$$

$$7^2 = x^2 + 9^2 - 2x(9) \cdot \cos(41^\circ)$$

$$49 = x^2 + 81 - (13.58)x$$

$$x^2 - (13.58)x + 32 = 0$$

We now use the quadratic formula with  $a = 1$ ,  $b = -13.58$ , and  $c = 32$ .

$$x = \frac{13.58 + \sqrt{(13.58)^2 - 4(32)}}{2} \approx 10.55 \text{ feet}$$

or

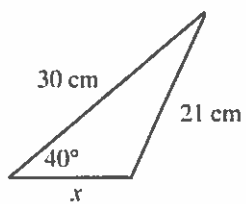
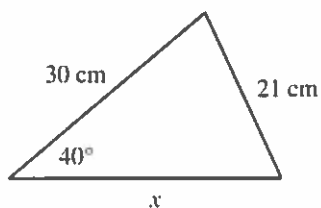
$$x = \frac{13.58 - \sqrt{(13.58)^2 - 4(32)}}{2} \approx 3.03 \text{ feet}$$

The side labeled  $x$  is either approximately 10.55 or 3.03 feet long.

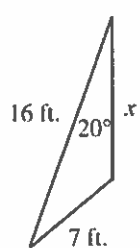
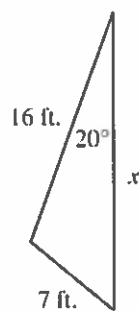
► Practice

Find the two possible lengths for the sides labeled  $x$ .

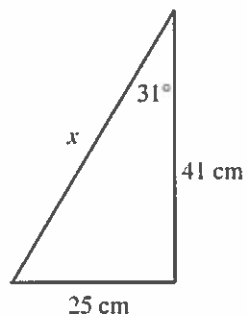
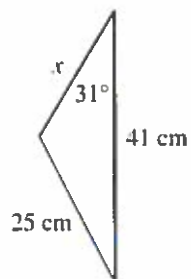
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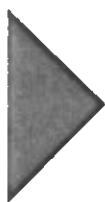
27.





L E S S O N

19



## Heron's Formula

**H**eron's formula enables us to calculate the area of a non-right triangle, just by knowing the lengths of the three sides. We prove the formula in this lesson and show how it can be used to find the areas of triangles and polygons.

The area of a triangle is  $\frac{1}{2}bh$ , where  $b$  is the length of the base and  $h$  is the height, as shown in Figure 19.1.

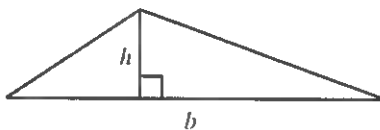


Figure 19.1

This is because the triangle has exactly half the area of a rectangle with the same base and height (shown in Figure 19.2).

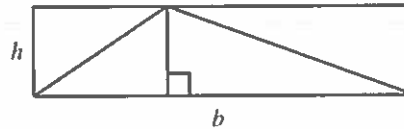


Figure 19.2

If we know the three sides of a triangle, we can find the height with some work. For example, suppose the three sides of a triangle are 4, 6, and 7 inches long. First, turn the triangle so the longest side is down, as shown in Figure 19.3.

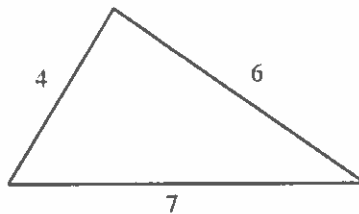


Figure 19.3

The height (sometimes called the **altitude**) will meet the longest side and divide it into two lengths. These can be called  $x$  and  $7 - x$ , as illustrated in Figure 19.4.

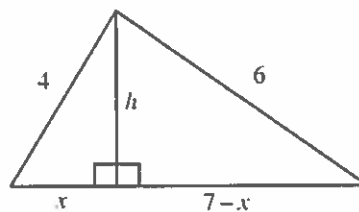


Figure 19.4

We now have two right triangles. If we apply the Pythagorean theorem to the one on the left, we get

$$x^2 + h^2 = 4^2$$

If we apply the Pythagorean theorem to the triangle on the right, we get

$$(7 - x)^2 + h^2 = 6^2$$

$$7^2 - 2(7)x + x^2 + h^2 = 6^2$$

We already know that  $x^2 + h^2 = 4^2$ , so we substitute this in:

$$7^2 - 2(7)x + (x^2 + h^2) = 6^2$$

$$7^2 - 2(7)x + 4^2 = 6^2$$

## HERON'S FORMULA

We can solve for  $x$ .

$$7^2 + 4^2 - 6^2 = 2(7)x$$

$$x = \frac{7^2 + 4^2 - 6^2}{2(7)}$$

Using  $x^2 + h^2 = 4^2$ , we can find the height of the triangle:

$$h = \sqrt{4^2 - x^2} = \sqrt{4^2 - \left(\frac{7^2 + 4^2 - 6^2}{2(7)}\right)^2} \approx 3.422$$

With the base  $b = 7$ , the triangle's area is thus

$$\text{area} = \frac{1}{2}bh = \frac{1}{2}(7)\sqrt{4^2 - \left(\frac{7^2 + 4^2 - 6^2}{2(7)}\right)^2} \approx \frac{1}{2}(7)(3.422) = 11.977 \text{ square inches}$$

In general, if the triangle's sides are  $a$ ,  $b$ , and  $c$ , then by the same computation as in the previous example, the area will be

$$\text{area} = \frac{1}{2}c\sqrt{a^2 - \left(\frac{c^2 + a^2 - b^2}{2c}\right)^2}$$

There is an easier formula, which is credited to Heron, who lived in Alexandria, Egypt, around 100 A.D. It uses the semiperimeter  $s$ , which is half of the perimeter.

$$s = \frac{1}{2}(a + b + c)$$

Using this, the area of a triangle with sides  $a$ ,  $b$ , and  $c$  is

$$\text{area} = \sqrt{s(s-a)(s-b)(s-c)}$$

In the example with sides 4, 6, and 7 inches, the semiperimeter is  $s = \frac{1}{2}(4 + 6 + 7) = 8.5$  and the area is

$$\text{area} = \sqrt{8.5(8.5-4)(8.5-6)(8.5-7)}$$

$$\text{area} = \sqrt{8.5(4.5)(2.5)(1.5)} \approx 11.977 \text{ square inches}$$

This is the same answer we obtained previously. If you replace each  $s$  in Heron's formula with  $s = \frac{1}{2}(a + b + c)$  and then multiply everything out, you would see that this is the same as the first formula. The algebra, however, would fill several pages, and so we omit doing this here.

**Example**

What is the area of the triangle with sides of length 10 feet, 15 feet, and 17 feet?

$$s = \frac{1}{2}(10 + 15 + 17) = 21$$

$$\text{area} = \sqrt{s(s-a)(s-b)(s-c)}$$

$$\text{area} = \sqrt{21(21-10)(21-15)(21-7)}$$

$$\text{area} = \sqrt{21(11)(6)(4)} = \sqrt{5,544} \approx 74.458 \text{ square feet}$$

**Example**

What is the area of an equilateral triangle with all sides 6 inches in length?

$$s = \frac{1}{2}(6 + 6 + 6) = 9$$

$$\text{area} = \sqrt{s(s-a)(s-b)(s-c)}$$

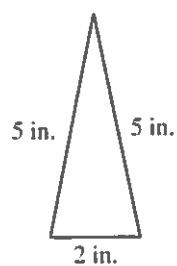
$$\text{area} = \sqrt{9(9-6)(9-6)(9-6)}$$

$$\text{area} = \sqrt{9(3)(3)(3)} = 9\sqrt{3} \approx 15.588 \text{ square inches}$$

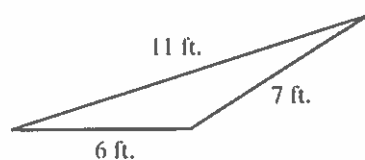
**► Practice**

Find the area of each triangle

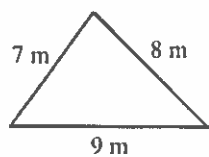
1.



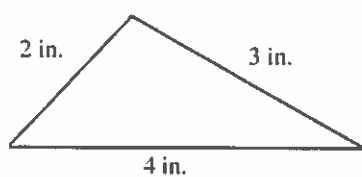
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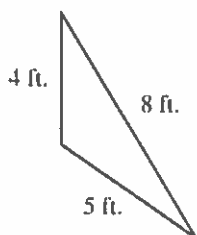
3.



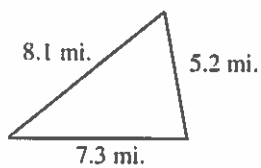
4.



5.



6.



Find the area of the triangle with edges of the three given lengths.

7. 6, 10, 12

8. 9, 9, 10

9. 4, 4, 4

10. 10, 14, 20

11. 6, 8, 9

12. 12, 15, 17

13. 30, 40, 55

14. 10, 16, 19

15. 33, 44, 55

16. 4, 12, 15

17. 2, 14, 15

18. 9.6, 12.2, 15.6

19. 31.8, 39.4, 40.2

20. 17.8, 18.2, 18.6

### ► Areas of Polygons

When trying to find the area of a figure made of straight lines, it helps to break it down into triangles.

#### Example

What is the area of the quadrilateral in Figure 19.11?

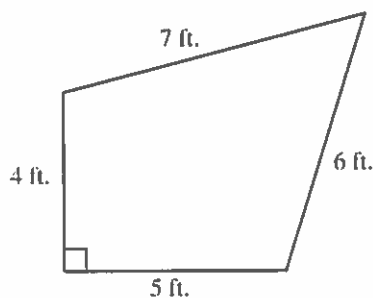


Figure 19.11

We divide this into two triangles with a diagonal line, as shown in Figure 19.12. Because one triangle is right, we can use the Pythagorean theorem to find the length  $d$  of the diagonal.

$$d^2 = 4^2 + 5^2$$

$$d = \sqrt{16 + 25} = \sqrt{41} \approx 6.4 \text{ feet}$$

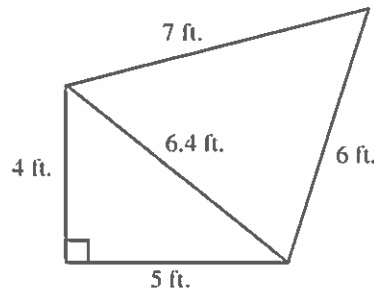


Figure 19.12

The right triangle has height 4 feet and base 5 feet, so its area is  $\frac{1}{2}(4)(5) = 10$  square feet.

The larger triangle has sides of length 6 feet, 7 feet, and around 6.4 feet, so its semiperimeter is  $s = \frac{1}{2}(6 + 7 + 6.4) = 9.7$ . Thus, we can use Heron's formula to find the area.

$$\text{area} = \sqrt{s(s-a)(s-b)(s-c)}$$

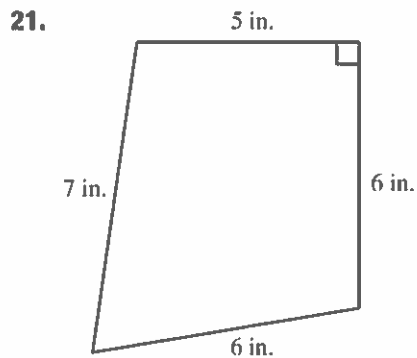
$$\text{area} = \sqrt{9.7(9.7-6)(9.7-7)(9.7-6.4)}$$

$$\text{area} = \sqrt{9.7(3.7)(2.7)(3.3)} \approx 17.88 \text{ square feet}$$

The area of the quadrilateral is thus  $10 + 17.88 = 27.88$  square feet.

### ► Practice

Find the area of the figure.

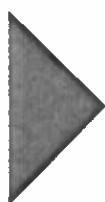






L E S S O N

20



## Advanced Problems

### ► Word Problems

At this point, we are now able to solve just about any problem that involves a triangle. The main trick is to draw the triangle and include all the information that is given. The word problems in this lesson will require knowledge and formulas from throughout this book.

#### **Example**

A car is driven down a straight desert highway. When the trip begins, a mesa off in the distance is  $20^\circ$  to the right. After the car has been driven for 45 miles, the mesa appears to be  $50^\circ$  to the right. At this point, how far away is the mesa?

All of our angles are relative to the direction of the highway, so we can draw it running from left to right as in Figure 20.1.

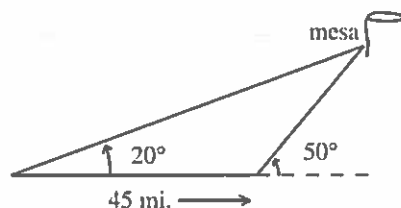


Figure 20.1

Thus, our task is to find the length  $x$  of the triangle in Figure 20.2.

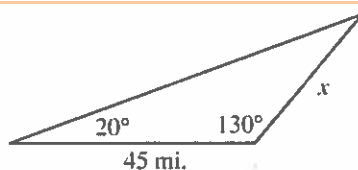


Figure 20.2

This can be solved by the Law of Sines as soon as we calculate the measure of the angle opposite the 45-mile side. This angle is  $180^\circ - 20^\circ - 130^\circ = 30^\circ$ . Thus, we have

$$\frac{\sin(a)}{A} = \frac{\sin(b)}{B}$$

$$\frac{\sin(30^\circ)}{45} = \frac{\sin(20^\circ)}{x}$$

$$x = \frac{45 \cdot \sin(20^\circ)}{\sin(30^\circ)} \approx 30.8 \text{ miles}$$

Thus, the mesa is about 30.8 miles away from the car after the 45 miles have been driven.

### Example

A spy watches the enemy launch a rocket straight upward. The spy is hidden 2 miles from the launch site. When the rocket first comes into sight, it is  $37^\circ$  above the horizontal. Two seconds later, it is  $43^\circ$  above the horizontal. How far did the rocket travel in those two seconds?

Figure 20.3 illustrates this problem.

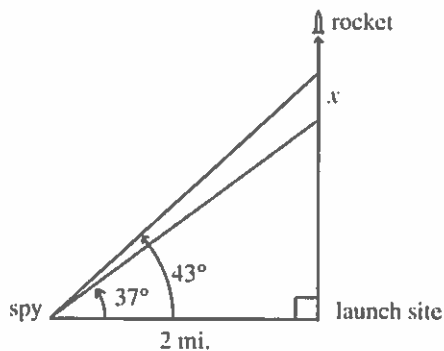


Figure 20.3

We have two right triangles, and thus, we can use basic trigonometric functions. The height  $h$  of the rocket when it was  $37^\circ$  above the horizontal can be calculated as

$$\tan(37^\circ) = \frac{h}{2}$$

$$h = 2 \cdot \tan(37^\circ) \approx 1.5 \text{ miles}$$

The height of the rocket when it was  $43^\circ$  above the horizontal is thus  $h + x$ , which is calculated by

$$\tan(43^\circ) = \frac{x + h}{2}$$

$$x + h = 2 \cdot \tan(43^\circ) \approx 1.87 \text{ miles}$$

Thus,

$$x = (x + h) - h \approx 1.87 - 1.5 = 0.37 \text{ miles}$$

In those two seconds, the rocket traveled 0.37 miles, or about 1,954 feet.

### Example

Suppose that everyone in town knows that a 30-foot-tall steeple has been built on top of a particularly large church. When you go to look at it, you notice that the tip of the steeple is  $38^\circ$  above the horizontal while the base of the steeple is  $32^\circ$  above the horizontal. How tall is the church together with the steeple?

Figure 20.4 illustrates the given information. We need to find the length  $x$  and then add the 30 feet for the steeple.

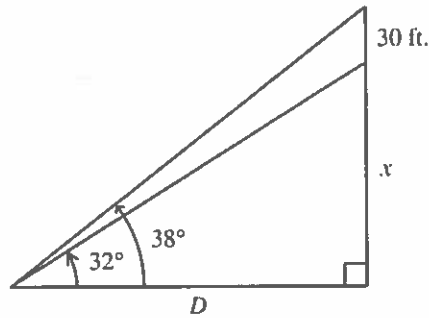


Figure 20.4

We will need to use the distance  $D$  to the church, which forms the base of the two right triangles in the figure. The first one has height  $x$  and angle  $32^\circ$ , thus

$$\tan(32^\circ) = \frac{x}{D}, \text{ so } D = \frac{x}{\tan(32^\circ)}$$

The larger right triangle has a height of  $x + 30$  and an angle of  $38^\circ$ , thus

$$\tan(38^\circ) = \frac{x + 30}{D}, \text{ so } D = \frac{x + 30}{\tan(38^\circ)}$$

If we put these two equations together, we get

$$\frac{x}{\tan(32^\circ)} = D = \frac{x + 30}{\tan(38^\circ)}$$

$$\frac{\tan(38^\circ) \cdot x}{\tan(32^\circ)} = x + 30$$

With a calculator, we estimate that  $\frac{\tan(38^\circ)}{\tan(32^\circ)} \approx 1.25$ , thus,

$$1.25x = x + 30$$

$$0.25x = 30$$

$$x = \frac{30}{0.25} = 120$$

Thus, the church is  $120 + 30 = 150$  feet tall with the steeple.

**Example**

What is the area of a pentagon with all sides 4 inches in length and all angles the same (a **regular pentagon**)?

With two diagonal lines, the pentagon can be split into three triangles, as shown in Figure 20.5. Each of these triangles has a sum of  $180^\circ$  worth of angles. Thus, the total sum of the angles of the pentagon is  $3 \cdot 180^\circ = 540^\circ$ . As all five of the angles are the same, each one measures  $\frac{540^\circ}{5} = 108^\circ$ .

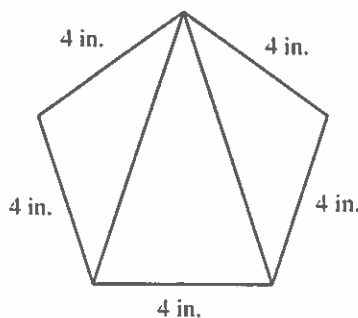


Figure 20.5

We could use trigonometry to find the lengths and then the areas of the three triangles in Figure 20.5. However, it is much easier to use the center of the pentagon to split it into triangles, as shown in Figure 20.6.

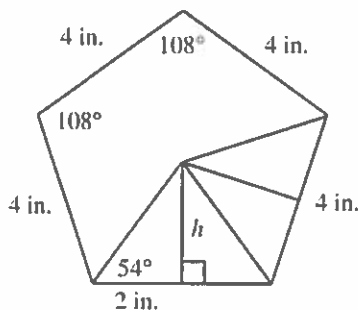


Figure 20.6

This divides the pentagon into 10 right triangles, each with base 2 inches, height  $h$ , and angle  $54^\circ$ . Thus,

$$\tan(54^\circ) = \frac{h}{2}$$

$$h = 2 \cdot \tan(54^\circ) \approx 2.75 \text{ inches}$$

The area of each of these 10 triangles is  $\frac{1}{2}b \cdot h = \frac{1}{2}(2)(2.75) = 2.75$  square inches. Thus, the area of the pentagon is about  $10(2.75) = 27.5$  square inches.

### Example

A marshy piece of property is triangular in shape. One angle measures  $34^\circ$ , and the two sides adjacent are 800 feet and 1,000 feet in length. What is the length of the third side? How many acres is the piece of property? (Note: An acre measures 43,560 square feet.)

The third side can be found directly by the Law of Cosines. If the third length is  $C$ , then

$$C^2 = A^2 + B^2 - 2AB \cos(\theta)$$

$$C^2 = 800^2 + 1,000^2 - 2(800)(1,000)\cos(34^\circ)$$

$$C = \sqrt{1,640,000 - 1,600,000 \cdot \cos(34^\circ)} \approx 560 \text{ feet}$$

The area can then be found by Heron's formula:

$$s = \frac{1}{2}(a + b + c) = \frac{1}{2}(800 + 1,000 + 560) = 1,180$$

$$\text{area} = \sqrt{1,180(1,180 - 1,000)(1,180 - 800)(1,180 - 560)}$$

$$\text{area} = \sqrt{1,180(180)(380)(620)} \approx 223,699 \text{ square feet, or about 5.13 acres}$$

### ► Practice

1. A motorcycle has two 14-inch-long pistons, which meet at a  $45^\circ$  angle at the bottom. How far apart are the tops of the pistons?
2. Denise wants to put mulch on a triangular flower bed that measures 12 feet by 10 feet by 8 feet. If she wants the mulch to be 4 inches thick, how many cubic feet of mulch will she need?
3. Rebecca needs a triangle with sides that measure 5 inches, 6 inches, and 7 inches for a project. What is the smallest angle of this triangle?
4. The flagpole on top of a building is 25 feet tall. If the angle from the ground to the top of the pole is  $55^\circ$  and the angle to the bottom of the pole is  $51^\circ$ , then how tall is the building (without counting the flagpole)?
5. A car is driven on a straight road for 25 miles. When the drive begins, a mountain in the distance is  $30^\circ$  to the right of the road. When the car stops after being driven the 25 miles, the mountain appears to be  $45^\circ$  to the right. How far away is the mountain from the car at this time?

### ADVANCED PROBLEMS

6. Marissa tap dances 20 feet straight across a stage. She then turns  $38^\circ$  and taps 12 feet further. How far is she now from where she started?
7. A ship spots a lighthouse at  $17^\circ$  north of east, then sails  $10^\circ$  west of north for 15 miles. The lighthouse is then  $2^\circ$  south of east. How far is the ship from the lighthouse at this point?
8. June, standing 100 feet away from a church, sees the tip of the steeple to be about  $51^\circ$  above the horizontal. The base of the steeple appears to be  $42^\circ$  above the horizontal. How tall is the steeple of the church?
9. Two straight antenna meet at a  $17^\circ$  angle. If one is 40 inches long and the other is 50 inches long, how far apart are the tips of the antenna?
10. Suppose you want to adjust the antenna from the last problem so that their ends are 25 inches apart. At what angle should they be set?
11. What is the area of a regular heptagon (a seven-sided figure with all angles the same) with each side 8 inches in length?
12. An insect's wing is in the shape of a triangle, measuring 10 mm by 17 mm by 15 mm. What is the surface area of the wing?
13. Andy sees a mountain to the north. He then hikes straight northeast for 3.4 miles. At this point, the mountain now appears to be  $23^\circ$  west of north. How far away is the mountain?
14. Two children, Bailey and Maddie, are playing together. Just for fun, they run apart at a  $120^\circ$  angle. Bailey runs straight for 70 feet while Maddie runs straight for 55 feet. When they stop, how far apart are they?
15. A triangular plot of land measures 398 feet by 471 feet by 503 feet. How many acres is this? (Remember that an acre is 43,560 square feet.)
16. Todd stands on the ground and looks up at an office building. The angle to the first window is  $42^\circ$ . The angle to the window above it is  $47^\circ$ . Assuming that the first window is 10 feet below the second, how far away is the building?
17. A team of explorers is trying to map a series of islands. As a guide, they use the highest point on each island. From one point on a beach, the mountaintop of an adjacent island appears to be  $27^\circ$  above the horizontal. From a second point, 500 feet directly back, the angle is  $25^\circ$ . How far away is the mountain (measured horizontally) from the first point on the beach?

### ADVANCED PROBLEMS

- 18.** The end of a 20-foot ladder rests on a railing. The end of a 12-foot ladder is right next to it. Suppose the longer ladder touches the ground 10 feet further away than the shorter ladder. How tall is the railing?
- 19.** Donald wants to cut a sail that is 17 feet by 12 feet by 8 feet. How many square feet of sailcloth will be needed? What will be the measure of the obtuse angle of the sail?
- 20.** Heather wants to cover a triangle with gold leaf. The three sides of the triangle are 40 inches by 35 inches by 37 inches. How many square inches of gold leaf will she need?
- 21.** Kelli has collected four triangles. The first is 4 inches by 6 inches by 7 inches. The second is 3 inches by 4 inches by 6 inches. The third is 8 inches by 10 inches by 13 inches. The last is 12 inches by 14 inches by 17 inches. Which of these triangles has the largest angle?



# Posttest



If you have completed all 20 lessons in this book, you are ready to take the posttest to measure your progress. The posttest has 50 multiple-choice questions covering the topics you studied in this book. Although the format of the posttest is similar to that of the pretest, the questions are different.

Take as much time as you need to complete the posttest. When you are finished, check your answers with the answer key that follows the posttest. Along with each answer is a number that tells you which lesson of this book teaches you about the trigonometry skills needed for that question. Once you know your score on the posttest, compare the results with the pretest. If you scored better on the posttest than you did on the pretest, congratulations! You have profited from your hard work. At this point, you should look at the questions you missed, if any. Do you know why you missed the question, or do you need to go back to the lesson and review the concept? If you can figure out why you missed the problem, then you understand the concept and simply need to concentrate more on accuracy when taking a test. If you missed a question because you did not know how to work the problem, go back to the lesson and spend more time working that type of problem. Take the time to understand trigonometry thoroughly. You need a solid foundation in trigonometry if you plan to use this information or progress to a higher level. Whatever your score on this posttest, keep this book for review and future reference.



► Answer Sheet

1. ☐ a ☐ b ☐ c ☐ d
2. ☐ a ☐ b ☐ c ☐ d
3. ☐ a ☐ b ☐ c ☐ d
4. ☐ a ☐ b ☐ c ☐ d
5. ☐ a ☐ b ☐ c ☐ d
6. ☐ a ☐ b ☐ c ☐ d
7. ☐ a ☐ b ☐ c ☐ d
8. ☐ a ☐ b ☐ c ☐ d
9. ☐ a ☐ b ☐ c ☐ d
10. ☐ a ☐ b ☐ c ☐ d
11. ☐ a ☐ b ☐ c ☐ d
12. ☐ a ☐ b ☐ c ☐ d
13. ☐ a ☐ b ☐ c ☐ d
14. ☐ a ☐ b ☐ c ☐ d
15. ☐ a ☐ b ☐ c ☐ d
16. ☐ a ☐ b ☐ c ☐ d
17. ☐ a ☐ b ☐ c ☐ d

18. ☐ a ☐ b ☐ c ☐ d
19. ☐ a ☐ b ☐ c ☐ d
20. ☐ a ☐ b ☐ c ☐ d
21. ☐ a ☐ b ☐ c ☐ d
22. ☐ a ☐ b ☐ c ☐ d
23. ☐ a ☐ b ☐ c ☐ d
24. ☐ a ☐ b ☐ c ☐ d
25. ☐ a ☐ b ☐ c ☐ d
26. ☐ a ☐ b ☐ c ☐ d
27. ☐ a ☐ b ☐ c ☐ d
28. ☐ a ☐ b ☐ c ☐ d
29. ☐ a ☐ b ☐ c ☐ d
30. ☐ a ☐ b ☐ c ☐ d
31. ☐ a ☐ b ☐ c ☐ d
32. ☐ a ☐ b ☐ c ☐ d
33. ☐ a ☐ b ☐ c ☐ d
34. ☐ a ☐ b ☐ c ☐ d

35. ☐ a ☐ b ☐ c ☐ d
36. ☐ a ☐ b ☐ c ☐ d
37. ☐ a ☐ b ☐ c ☐ d
38. ☐ a ☐ b ☐ c ☐ d
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41. ☐ a ☐ b ☐ c ☐ d
42. ☐ a ☐ b ☐ c ☐ d
43. ☐ a ☐ b ☐ c ☐ d
44. ☐ a ☐ b ☐ c ☐ d
45. ☐ a ☐ b ☐ c ☐ d
46. ☐ a ☐ b ☐ c ☐ d
47. ☐ a ☐ b ☐ c ☐ d
48. ☐ a ☐ b ☐ c ☐ d
49. ☐ a ☐ b ☐ c ☐ d
50. ☐ a ☐ b ☐ c ☐ d



## ► Posttest

Solve questions 1–29 without using a calculator.

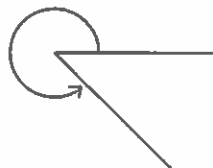
1. Convert the angle measure  $\frac{\pi}{4}$  into degrees.

- a.  $90^\circ$
- b.  $45^\circ$
- c.  $30^\circ$
- d.  $60^\circ$

2. Convert  $120^\circ$  into radians.

- a.  $120\pi$
- b.  $\frac{\pi}{12}$
- c.  $\frac{2\pi}{3}$
- d.  $\frac{3\pi}{4}$

3. Which of the following could be the measure of this angle?

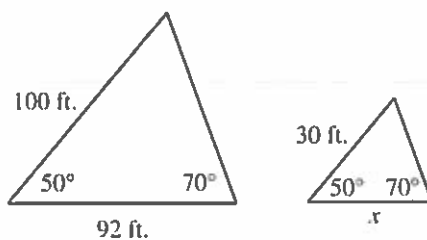


- a.  $135^\circ$
- b.  $225^\circ$
- c.  $270^\circ$
- d.  $315^\circ$

4. If a triangle has angles of measure  $24^\circ$  and  $68^\circ$ , then what is the measure of the third angle?

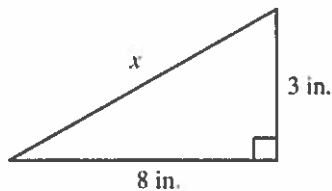
- a.  $46^\circ$
- b.  $78^\circ$
- c.  $88^\circ$
- d.  $92^\circ$

5. Find the length of the side marked  $x$ .



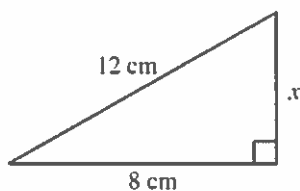
- a. 22 feet
- b. 26 feet
- c. 27.6 feet
- d. 30.7 feet

6. Find the length of the side labeled  $x$ .



- a.  $\sqrt{55}$  inches
- b.  $\sqrt{73}$  inches
- c. 11 inches
- d. 73 inches

7. Find the length of the side labeled  $x$ .



- a.  $4\sqrt{5}$  centimeters
- b.  $4\sqrt{10}$  centimeters
- c.  $4\sqrt{13}$  centimeters
- d. 10 centimeters

8. If  $g(x) = \frac{x+3}{x^2+2}$ , what is  $g(-1)$ ?

a.  $-2$

b.  $2$

c.  $\frac{2}{3}$

d.  $-\frac{x+3}{x^2+2}$

9. What is the domain of  $y(x) = \sqrt{4-x}$ ?

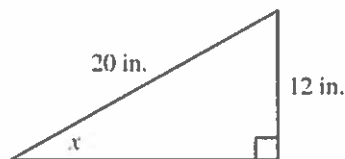
a.  $x < 4$

b.  $x \leq 4$

c.  $x \geq -4$

d.  $x > -4$

10. What is  $\sin(x)$  for angle  $x$  in the figure?



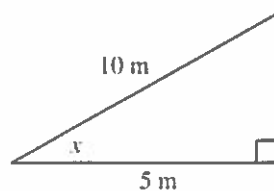
a.  $\frac{2}{5}$

b.  $\frac{3}{5}$

c.  $\frac{4}{5}$

d.  $16$

11. What is  $\sin(x)$  for angle  $x$  in the figure?



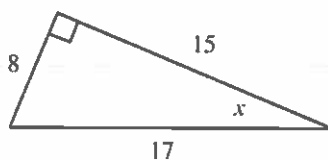
a.  $\frac{1}{2}$

b.  $\sqrt{3}$

c.  $5\sqrt{3}$

d.  $\frac{\sqrt{3}}{2}$

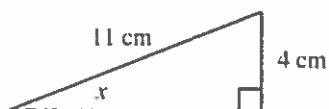
12. Find  $\cos(x)$  for the angle  $x$  in the figure.



- a.  $\frac{8}{17}$
  - b.  $\frac{8}{15}$
  - c.  $\frac{15}{17}$
  - d.  $\frac{17}{15}$
13. If  $\sin(x) = \frac{7}{10}$ , then what is  $\cos(x)$ ?

- a.  $\frac{3}{10}$
- b.  $\frac{10}{7}$
- c.  $\frac{\sqrt{51}}{7}$
- d.  $\frac{\sqrt{51}}{10}$

14. Find  $\tan(x)$  for angle  $x$  in the figure.



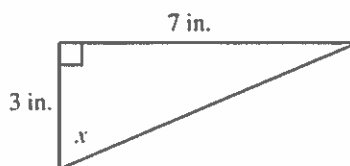
- a.  $\frac{\sqrt{105}}{4}$
- b.  $\frac{4\sqrt{105}}{105}$
- c.  $\frac{\sqrt{105}}{11}$
- d.  $\frac{4\sqrt{137}}{137}$



15. If  $\tan(x) = 2$ , then what is  $\cos(x)$ ?

- a.  $\frac{\sqrt{5}}{5}$
- b.  $\frac{2\sqrt{5}}{5}$
- c.  $\sqrt{5}$
- d.  $\frac{1}{2}$

Use the following to answer questions 16 and 17.



16. What is  $\sec(x)$  for the angle  $x$  in the figure?

- a.  $\frac{\sqrt{58}}{3}$
- b.  $\frac{\sqrt{58}}{7}$
- c.  $\frac{3\sqrt{58}}{58}$
- d.  $\frac{7\sqrt{58}}{58}$

17. What is  $\cot(x)$  for the angle  $x$  in the figure?

- a.  $\frac{3}{7}$
- b.  $\frac{7}{3}$
- c.  $\frac{\sqrt{58}}{3}$
- d.  $\frac{3\sqrt{58}}{58}$

18. Which of the following is NOT true?

- a.  $\cot(x) = \tan(90^\circ - x)$
- b.  $\csc(x) = \frac{1}{\sec(x)}$
- c.  $\sin^2(x) + \cos^2(x) = 1$
- d.  $\tan^2(x) + 1 = \sec^2(x)$

19. Write  $\frac{\csc(x)}{\sec(x)}$  entirely in terms of  $\sin(x)$  and  $\cos(x)$ .

a.  $\frac{1}{\cos^2(x)}$

b.  $\frac{1}{\sin(x)\cos(x)}$

c.  $\frac{\sin(x)}{\cos(x)}$

d.  $\frac{\cos(x)}{\sin(x)}$

20. Evaluate  $\tan(45^\circ)$ .

a.  $\frac{\sqrt{3}}{2}$

b.  $\frac{\sqrt{3}}{3}$

c. 1

d.  $\sqrt{3}$

21. Evaluate  $\cos\left(\frac{\pi}{6}\right)$ .

a. 0

b.  $\frac{1}{2}$

c.  $\frac{\sqrt{2}}{2}$

d.  $\frac{\sqrt{3}}{2}$

22. Evaluate  $\sec(60^\circ)$ .

a. 2

b.  $\frac{\sqrt{2}}{2}$

c.  $\frac{\sqrt{3}}{2}$

d.  $\frac{2\sqrt{3}}{3}$

**23.** At what point is the unit circle intersected by an angle of  $210^\circ$ ? (Suppose that the angle is measured counterclockwise from the positive  $x$ -axis.)

- a.  $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
- b.  $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
- c.  $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
- d.  $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

**24.** What is  $\cos(150^\circ)$ ?

- a.  $\frac{1}{2}$
- b.  $-\frac{1}{2}$
- c.  $\frac{\sqrt{3}}{2}$
- d.  $-\frac{\sqrt{3}}{2}$

**25.** What is  $\sin\left(\frac{7\pi}{4}\right)$ ?

- a.  $-\frac{\sqrt{2}}{2}$
- b.  $\frac{\sqrt{2}}{2}$
- c.  $-\frac{\sqrt{3}}{2}$
- d.  $\frac{\sqrt{3}}{2}$

**26.** Evaluate  $\tan\left(\frac{2\pi}{3}\right)$ .

- a. 1
- b. -1
- c.  $\sqrt{3}$
- d.  $-\sqrt{3}$

27. Evaluate  $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$ .

- a.  $30^\circ$
- b.  $45^\circ$
- c.  $60^\circ$
- d.  $90^\circ$

28. Evaluate  $\cos^{-1}(0)$ .

- a.  $30^\circ$
- b.  $45^\circ$
- c.  $60^\circ$
- d.  $90^\circ$

29. What is the area of a triangle with two sides 10 inches long and one side 6 inches long?

- a.  $25 \text{ in.}^2$
- b.  $30 \text{ in.}^2$
- c.  $3\sqrt{7} \text{ in.}^2$
- d.  $3\sqrt{91} \text{ in.}^2$

Use a calculator to solve questions 30–50.

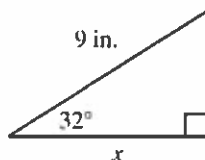
30. If a television screen is 28 inches on the diagonal and 24 inches wide, how tall is it?

- a. 14.4 in.
- b. 16.1 in.
- c. 27.5 in.
- d. 36.9 in.

31. If  $\cos(40^\circ) = 0.766$ , then what is  $\sin(50^\circ)$ ?

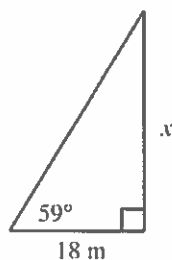
- a. 0.234
- b. 0.6128
- c. 0.766
- d. 0.9575

32. Use a calculator to estimate the length  $x$  in the figure.

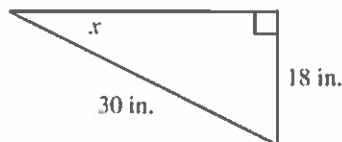


- a. 4.77 in.
- b. 7.51 in.
- c. 7.63 in.
- d. 10.61 in.

33. Use a calculator to estimate the length  $x$  in the figure.

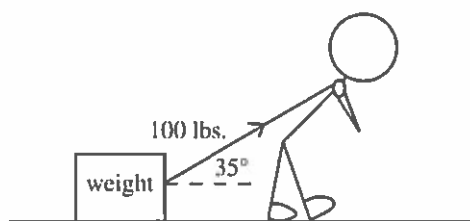


- a. 10.82 m
  - b. 14.86 m
  - c. 15.43 m
  - d. 29.96 m
34. Suppose a 25-foot ladder is leaned against a wall so that it makes a  $70^\circ$  angle with the ground. How far from the wall is the base of the ladder?
- a. 8.6 ft.
  - b. 23.5 ft.
  - c. 26.6 ft.
  - d. 73 ft.
35. A wire runs straight from the top of a 40-foot pole to the ground. If it will make a  $41^\circ$  angle with the ground, how long is the wire?
- a. 26 ft.
  - b. 46 ft.
  - c. 53 ft.
  - d. 61 ft.
36. Use a calculator to estimate the angle  $x$  in the figure.



- a.  $31^\circ$
- b.  $36.9^\circ$
- c.  $53.1^\circ$
- d.  $59^\circ$

- 37.** A 40-foot rope is attached to a window that is 25 feet off the ground. If the rope is pulled tight to a spot on the ground, what angle will it make with the ground?
- $32^\circ$
  - $39^\circ$
  - $51^\circ$
  - $58^\circ$
- 38.** A ramp will run 25 feet horizontally and 5 feet vertically. At what angle will the ramp run?
- $11.3^\circ$
  - $11.5^\circ$
  - $12.8^\circ$
  - $15.1^\circ$
- 39.** Suppose a 10-foot board leans against a tree. If the bottom of the board is 4 feet from the tree, what angle does the board make with the ground?
- $21.8^\circ$
  - $23.6^\circ$
  - $51.8^\circ$
  - $66.4^\circ$
- 40.** Suppose a person drags a heavy weight across the ground by pulling with 100 pounds of force at a  $35^\circ$  angle, as illustrated in the figure. How much of the force is actually pulling the weight in the direction it is moving?

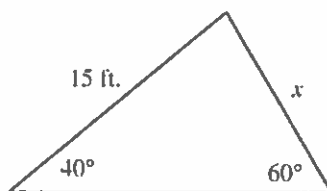


- 57 lbs.
- 65 lbs.
- 70 lbs.
- 82 lbs.

**41.** A ship travels 15 miles in the direction  $20^\circ$  west of south, then 8 miles at  $15^\circ$  north of west. In what direction should it head to go straight back to where it began?

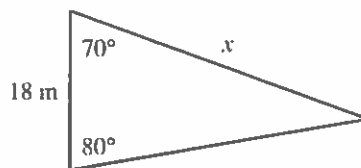
- a.  $28^\circ$  north of east
- b.  $37^\circ$  north of east
- c.  $43^\circ$  north of east
- d.  $41^\circ$  east of north

**42.** Find the length  $x$  in the figure.



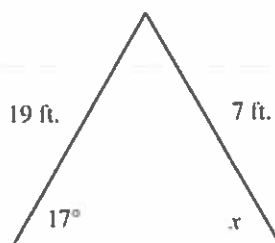
- a. 9.6 ft.
- b. 11.1 ft.
- c. 13.2 ft.
- d. 20.2 ft.

**43.** Find the length  $x$  in the figure.

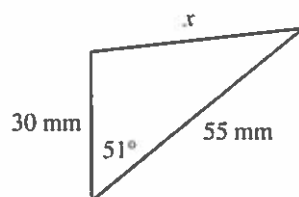


- a. 29 m
- b. 31 m
- c. 32 m
- d. 35 m

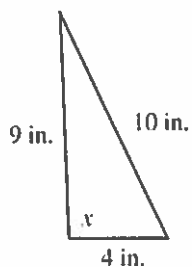
44. The following figure is not drawn accurately. Find the two possible measurements for angle  $x$ .



- a.  $52.5^\circ$  and  $127.5^\circ$
  - b.  $54.1^\circ$  and  $125.9^\circ$
  - c.  $57.2^\circ$  and  $122.8^\circ$
  - d.  $62.5^\circ$  and  $117.5^\circ$
45. Find the length  $x$  in the figure.



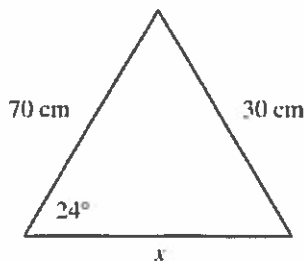
- a. 38 mm
  - b. 40 mm
  - c. 43 mm
  - d. 47 mm
46. Find the measure of angle  $x$  in the figure.



- a.  $91.7^\circ$
- b.  $92.4^\circ$
- c.  $92.9^\circ$
- d.  $93.2^\circ$



47. What are the two lengths that  $x$  could be in the figure (not drawn to scale)?



- a. 53.2 cm or 74.5 cm
  - b. 54.5 cm or 73.4 cm
  - c. 56.2 cm or 71.5 cm
  - d. 57.1 cm or 70.4 cm
48. What is the approximate area of a triangle with sides of length 20 feet, 24 feet, and 25 feet?
- a.  $223 \text{ ft.}^2$
  - b.  $238 \text{ ft.}^2$
  - c.  $240 \text{ ft.}^2$
  - d.  $247 \text{ ft.}^2$
49. A car drives straight through the desert for 8 miles. At first, a distant rock formation appears to be  $23^\circ$  to the left of the road. After driving the 8 miles, the rocks appear to be  $45^\circ$  to the left of the road. How far away is the rock formation at this time?
- a. 6.2 mi.
  - b. 6.9 mi.
  - c. 7.8 mi.
  - d. 8.3 mi.
50. What is the area of a pentagon with five sides, all 6 inches long? (Assume all angles are of the same measure.)
- a.  $30 \text{ in.}^2$
  - b.  $36 \text{ in.}^2$
  - c.  $62 \text{ in.}^2$
  - d.  $72 \text{ in.}^2$

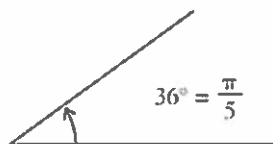


# Answer Key



## ► Lesson 1

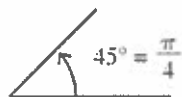
1.  $36^\circ = \frac{\pi}{5}$



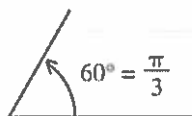
2.  $120^\circ = \frac{2\pi}{3}$



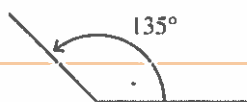
3.  $45^\circ = \frac{\pi}{4}$



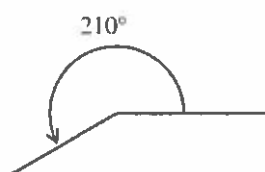
4.  $60^\circ = \frac{\pi}{3}$



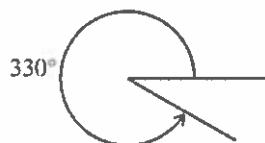
5.  $\frac{3\pi}{4}$



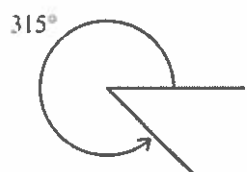
6.  $\frac{7\pi}{6}$



7.  $\frac{11\pi}{6}$



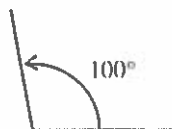
8.  $\frac{7\pi}{4}$



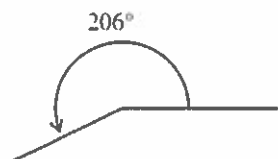
9.  $\frac{\pi}{9}$



10.  $\frac{5\pi}{9}$



11.  $\frac{103\pi}{90}$



12.  $\frac{4\pi}{5}$



13.  $\frac{\pi}{180}$



14.  $225^\circ$

15.  $240^\circ$

16.  $300^\circ$

17.  $22.5^\circ$

18.  $108^\circ$

19.  $126^\circ$

20.  $\frac{120^\circ}{\pi}$

21.  $\frac{720^\circ}{\pi}$

## ► Lesson 2

1.  $105^\circ$
2.  $46^\circ$
3.  $20^\circ$
4.  $\frac{5\pi}{12}$
5.  $\frac{7\pi}{30}$
6.  $\frac{13\pi}{30}$
7.  $\alpha = 60^\circ$
8.  $\alpha = 45^\circ$
9.  $\alpha = 75^\circ$
10.  $\alpha = \frac{\pi}{6}$
11.  $\alpha = \frac{\pi}{12}$
12.  $\alpha = \frac{\pi}{5}$
13. No, the triangles are not similar.
14. Yes, the triangles are similar.
15. No. The first triangle has angles  $40^\circ$ ,  $100^\circ$ , and  $40^\circ$ , while the second has angles  $50^\circ$ ,  $100^\circ$ , and  $30^\circ$ .
16. Yes, the last two angles of the first triangle have the same measure  $\theta$  because the triangle is isosceles. Thus,  $\theta + \theta + \frac{\pi}{6} = \pi$ , so  $\theta = \frac{5\pi}{12}$ . Because you have seen that two of the corresponding angles are equal, you know that the triangles are similar.
17.  $x = 50$

18.  $x = 24$

19.  $x = \frac{40}{3} = 13\frac{1}{3}$

20.  $x = 42$

21.  $x = \frac{45}{4} = 11\frac{1}{4}$

22. This triangle is similar to any other equilateral triangle, which have all sides of the same length like this one,  $x = 10$

23. First we figure that the third side of the second triangle is 26. Then we figure that the third triangle's sides are  $k$  times those of the second, since they are similar.

$$x = k \cdot 11$$

$$40 = k \cdot 26$$

$$k = \frac{40}{26} = \frac{20}{13}$$

$$x = \frac{220}{13} = 16\frac{12}{13}$$

### ► Lesson 3

1.  $x = 5$  ft.

2.  $x = 12$  in.

3.  $x = \sqrt{300} = 10\sqrt{3} \approx 17.32$  m

4.  $x = \sqrt{85} \approx 9.22$  cm

5.  $x = \sqrt{161} \approx 12.69$  in.

6.  $x = \sqrt{32} = 4\sqrt{2} \approx 5.66$  ft.

7.  $x = \sqrt{14.43} \approx 3.8$  m

8.  $x = 73\frac{7}{36} \approx 73.19$  ft.

9.  $\sqrt{53} \approx 7.28$  ft.
10.  $\sqrt{75} = 5\sqrt{3} \approx 8.66$  ft.
11. 40 cm
12.  $\sqrt{5} \approx 2.24$  m
13.  $\sqrt{32} = 4\sqrt{2} \approx 5.66$  mi.
14.  $\sqrt{72} = 6\sqrt{2} \approx 8.49$  in.
15.  $\sqrt{200} = 10\sqrt{2} \approx 14.14$  in.
16.  $\sqrt{11.52} \approx 3.39$  m
17.  $\sqrt{817.19} \approx 28.59$  in.
18.  $\sqrt{54.5} \approx 7.38$  ft.
19.  $\sqrt{193.25} \approx 13.9$  in.
20.  $\sqrt{375} = 5\sqrt{15} \approx 19.36$  ft.
21.  $\sqrt{52} \approx 7.2$  in.
22.  $\sqrt{800} = 20\sqrt{2} \approx 28.28$  in.
23. a Pythagorean triple
24. not a Pythagorean triple
25. not a Pythagorean triple
26. a Pythagorean triple
27. a Pythagorean triple
28. 8-15-17



29. 12-5-13, although it is usually written as 5-12-13

30. 65-72-97

31. 16-30-34

32. 60-91-109

## ► Lesson 4

1.  $y(3) = 17$ , so 3 relates to 17

2.  $V(4) = 64$

3.  $g(4) = 9$ , so 4 relates to 9

4. Because the variable  $x$  doesn't appear in the formula, there is no place to plug in  $x = 17$ . Thus,  $f(17) = 10$ .

5.  $s(-1) = 80$

6.  $C(2.7) = 5.4\pi$

7.  $k(2) = -9$

8.  $-1$  relates to  $q(-1) = 0$

9.  $f\left(\frac{1}{2}\right) = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$

10.  $p(-2) = -\frac{4}{3}$ , so  $p$  relates  $-2$  to  $-\frac{4}{3}$

11.  $h(10) = \sqrt{34}$ , so  $h$  takes 10 to  $\sqrt{34}$

12.  $h(-7) = -1$

13.  $k\left(\frac{1}{3}\right) = -\frac{5}{4}$

14.  $k(3) = \frac{4}{5}$ , so  $k$  takes 3 to  $\frac{4}{5}$

15.  $x \neq -5$

16. the domain of  $g$  consists of all real numbers,  $\mathbb{R}$

17.  $x \geq -3$

18.  $x \geq -\frac{2}{5}$

19.  $x \neq -2$  and  $x \neq -4$

20.  $x \leq 4$

21.  $x \geq -5$ , except  $x \neq 2$  and  $x \neq -2$

22.  $x \leq \frac{3}{4}$

23.  $x \leq 3$

24.  $x \leq 2$ , except  $x \neq 0, 1, -5$

25. While any number can be plugged into  $A(x) = \frac{1}{2}x^2$ , we must consider the fact that  $x$  represents the base and height of a triangle. This means that  $x$  must be a positive amount. Thus, the domain of this function consists of all positive lengths  $x > 0$ .

26. The domain of  $r$  consists of all amounts of time  $t > 0$ . Negative amounts of time are impossible.

27.  $h(207.32) = 3$

28.  $h(39.5) = 2$

29.  $k(3,185) = 8,185$

30.  $k(13.843) = 18.848$

## ► Lesson 5

1.  $\sin(x) = \frac{5}{13}$

2.  $\sin(x) = \frac{15}{17}$

3.  $\sin(x) = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$

4.  $\sin(x) = \frac{\sqrt{5}}{\sqrt{11}} = \frac{\sqrt{55}}{11}$

5.  $\sin(x) = \frac{\sqrt{19}}{10}$

6.  $\sin(x) = \frac{\sqrt{105}}{11}$

7.  $\sin(x) = \frac{1}{2}$

8.  $\sin(x) = \frac{5}{\sqrt{29}} = \frac{5\sqrt{29}}{29}$

9.  $\sin(x) = \frac{2}{5}$

10.  $\sin(x) = \frac{2}{4.47} \approx 0.45$

11.  $\sin(x) = \frac{5}{7}$

12.  $\sin(x) = \frac{4}{5}$

13.  $\sin(x) = \frac{5}{\sqrt{61}} = \frac{5\sqrt{61}}{61}$

14.  $\sin(x) = \frac{2}{9}$

$$15. \sin(x) = \frac{\sqrt{3}}{2}$$

$$16. \sin(x) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$17. \sin(x) = \frac{2\sqrt{10}}{7}$$

$$18. \sin(x) = \frac{2.3}{\sqrt{7.85}} \approx 0.82$$

$$19. \sin(x) = \frac{5}{6}$$

$$20. \sin(x) = \frac{\sqrt{66.15}}{9.2} \approx 0.88$$

$$21. O = 8 \text{ ft.}$$

$$22. H = 20 \text{ in.}$$

$$23. A = \sqrt{875} \approx 29.58 \text{ ft.}$$

$$24. A = \sqrt{924} \approx 30.4 \text{ in.}$$

$$25. O = 6 \text{ in.}$$

## ► Lesson 6

$$1. \cos(\theta) = \frac{4}{5}$$

$$2. \cos(\theta) = \frac{4}{7}$$

$$3. \cos(\theta) = \frac{3}{10}$$

$$4. \cos(\theta) = \frac{2}{13}$$

$$5. \cos(\theta) = \frac{6.5}{8.5} = \frac{65}{85} = \frac{13}{17}$$

$$6. \cos(\theta) = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$7. \cos(\theta) = \frac{\sqrt{23}}{12}$$

$$8. \cos(\theta) = \frac{3}{\sqrt{14}} = \frac{3\sqrt{14}}{14}$$

$$9. \cos(\theta) = \frac{8}{4\sqrt{5}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

$$10. \cos(\theta) = \frac{1}{2}$$

$$11. \cos(90^\circ - x) = \frac{2}{5}$$

$$12. \cos(x) = \frac{\sqrt{21}}{5}$$

$$13. \cos(60^\circ) = \frac{1}{2}$$

$$14. \cos(30^\circ) = \frac{\sqrt{3}}{2}$$

$$15. \cos(x) = \frac{3\sqrt{11}}{10}$$

$$16. \sin(50^\circ) \approx 0.766$$

$$17. \sin(x) = \frac{\sqrt{7}}{4}$$

$$18. \sin(x) = \frac{3\sqrt{5}}{7}$$

$$19. \sin(45^\circ) = \frac{\sqrt{2}}{2}$$

$$20. \sin(x) = \frac{\sqrt{14}}{4}$$

$$21. A = 1.54 \text{ ft. long}$$

22.  $H = 6$  ft. long

23.  $A = \frac{20}{7}$  in. long and  $O \approx 9.58$  in. long

### ► Lesson 7

1.  $\tan(x) = \frac{1}{2}$

2.  $\tan(x) = \frac{5}{12}$

3.  $\tan(x) = \frac{7}{4}$

4.  $\tan(x) = \frac{1}{3}$

5.  $\tan(x) = \frac{5}{\sqrt{200}} = \frac{\sqrt{2}}{4}$

6.  $\tan(x) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

7.  $\tan(x) = \frac{\sqrt{5}}{2}$

8.  $\tan(x) = \frac{5}{2}$

9.  $\tan(x) = \frac{\sqrt{21}}{2}$

10.  $\tan(x) = \frac{\sqrt{34.39}}{8.1} \approx 0.724$

11.  $\tan(x) = \frac{3}{4}$

12.  $\tan(x) = \frac{1}{2}$

13.  $\tan(x) = \frac{8}{15}$

14.  $\tan(x) = 4$

15.  $\tan(x) \approx 1.14$

16.  $\tan(x) = 1$

17.  $\sin(x) = \frac{2\sqrt{13}}{13}$

18.  $\sin(x) = \frac{3}{10}$

19.  $\cos(x) = \frac{\sqrt{33}}{7}$

20.  $\cos(\theta) = \frac{2}{5}$

21.  $\tan(x) = \frac{45}{28}$

22.  $\tan(x) = \frac{3\sqrt{5}}{2}$

23.  $\sin(x) = \frac{\sqrt{39}}{8}$

24.  $\cos(x) = \frac{7}{10}$

25.  $\cos(x) = \frac{\sqrt{14}}{5}$

26.  $\tan(x) = \frac{12}{7}$

27.  $\cos(x) = \frac{\sqrt{41}}{7}$

### ► Lesson 8

1.  $\sec(x) = \frac{5}{4}$

2.  $\csc(x) = \frac{5}{3}$

3.  $\cot(x) = \frac{4}{3}$

4.  $\cot(y) = \frac{2}{7}$

5.  $\sec(y) = \frac{\sqrt{53}}{2}$

6.  $\csc(y) = \frac{\sqrt{53}}{7}$

7.  $\cot(z) = \sqrt{3}$

8.  $\tan(w) = \sqrt{3}$

9.  $\sec(z) = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

10.  $\csc(w) = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

11.  $\frac{\sin(x)}{\cos^2(x)}$

12.  $\cos(x)$

13.  $\cos(x) \cdot \sin(x)$

14.  $\frac{1}{\cos(x)}$

15.  $\frac{1}{\sin^3(x) \cdot \cos(x)}$

16.  $\frac{1}{\sin(x)}$

17.  $\frac{\sin(x)}{\cos(x)}$

18.  $\frac{\sin^2(x)}{\cos^2(x)}$

19. 1

20.  $\sin(x)$

21.  $\sin(x) = \frac{\sqrt{7}}{4}$



22.  $\csc(x) = \sqrt{17}$

23.  $\sec(x) = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

24.  $\cot(x) = \frac{\sqrt{3}}{\sqrt{6}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

25. The sine is  $\frac{5}{6}$ .

26.  $\sec(x) = \frac{\sqrt{73}}{3}$

27.  $\cot(x) = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

### ► Lesson 9

1.  $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

2. 1

3. 1

4.  $\sqrt{2}$

5.  $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

6.  $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

7.  $\cos(60^\circ) = \frac{1}{2}$

8.  $\tan(60^\circ) = \sqrt{3}$

9.  $\sin(30^\circ) = \frac{1}{2}$

10.  $\cot(60^\circ) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

$$11. \csc\left(\frac{\pi}{3}\right) = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$12. \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

$$13. \sec\left(\frac{\pi}{6}\right) = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$14. \cos(30^\circ) = \frac{\sqrt{3}}{2}$$

$$15. \csc(60^\circ) = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$16. \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$17. \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$$

$$18. \cot\left(\frac{\pi}{3}\right) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$19. \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

$$20. \csc\left(\frac{\pi}{6}\right) = 2$$

$$21. \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$22. \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$23. \cot\left(\frac{\pi}{6}\right) = \sqrt{3}$$

$$24. \sec(30^\circ) = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

## ► Lesson 10

1.  $180^\circ$ , or  $\pi$  radians

2.  $-90^\circ$ , or  $-\frac{\pi}{2}$  radians

3.  $(0,1)$

4.  $(-1,0)$

5.  $(0,1)$

6.  $(1,0)$

7.  $(0,-1)$

8.  $(0,-1)$

9.  $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

10.  $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

11.  $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

12.  $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

13.  $(0,-1)$

14.  $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

15.  $(-1,0)$

16.  $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

17.  $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

18.  $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

19.  $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

20.  $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

21.  $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

22.  $(0,1)$

23.  $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

24.  $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

25.  $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

26.  $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

27.  $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

28.  $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

29.  $(0,1)$

### ► Lesson 11

1.  $\sin(45^\circ) = \frac{\sqrt{2}}{2}$  and  $\cos(45^\circ) = \frac{\sqrt{2}}{2}$

2.  $\sin(180^\circ) = 0$  and  $\cos(180^\circ) = -1$

3.  $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$  and  $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

4.  $\sin\left(-\frac{\pi}{2}\right) = -1$  and  $\cos\left(-\frac{\pi}{2}\right) = 0$

5.  $\sin(135^\circ) = \frac{\sqrt{2}}{2}$  and  $\cos(135^\circ) = -\frac{\sqrt{2}}{2}$

6.  $\sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$  and  $\cos\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

7.  $\sin(390^\circ) = \frac{1}{2}$  and  $\cos(390^\circ) = \frac{\sqrt{3}}{2}$

8.  $\sin(-60^\circ) = -\frac{\sqrt{3}}{2}$  and  $\cos(-60^\circ) = \frac{1}{2}$

9.  $\sin\left(-\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2}$  and  $\cos\left(-\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2}$

10.  $\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$  and  $\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

11.  $\sin(0^\circ) = 0$  and  $\cos(0^\circ) = 1$

12.  $\sin(\pi) = 0$  and  $\cos(\pi) = -1$

13.  $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$  and  $\cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

14.  $\sin\left(-\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2}$  and  $\cos\left(-\frac{2\pi}{3}\right) = -\frac{1}{2}$

15.  $\sin(330^\circ) = -\frac{1}{2}$  and  $\cos(330^\circ) = \frac{\sqrt{3}}{2}$

16.  $\sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}$  and  $\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

17.  $\sec(60^\circ) = 2$

18.  $\csc(-30^\circ) = -2$

19.  $\sec\left(-\frac{\pi}{6}\right) = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

20.  $\tan(0) = 0$

21.  $\cot(150^\circ) = -\sqrt{3}$

22.  $\csc(120^\circ) = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

23.  $\sec(-45^\circ) = \sqrt{2}$

24.  $\tan\left(\frac{\pi}{4}\right) = 1$

25.  $\cot\left(\frac{\pi}{6}\right) = \sqrt{3}$

26.  $\tan(240^\circ) = \sqrt{3}$

27.  $\tan\left(-\frac{\pi}{3}\right) = -\sqrt{3}$

28.  $\cot(180^\circ)$  is undefined

29.  $\csc(225^\circ) = -\sqrt{2}$

## ► Lesson 12

1.  $\sin(50^\circ) \approx 0.766$

2.  $\cos(30^\circ) \approx 0.866$

3.  $\tan(85^\circ) \approx 11.430$

4.  $\sin(315^\circ) \approx -0.707$

5.  $\cos(-80^\circ) \approx 0.174$

6.  $\tan(14^\circ) \approx 0.249$

7.  $\sin\left(\frac{\pi}{5}\right) \approx 0.588$

8.  $\cos(\pi) = -1$

9.  $\tan\left(\frac{\pi}{4}\right) = 1$

10.  $\sin\left(-\frac{\pi}{10}\right) \approx -0.309$

11.  $\cos(8.63) \approx -0.700$

12.  $\tan\left(\frac{\pi}{3}\right) \approx 1.732$

13.  $\sec(18^\circ) \approx 1.051$

14.  $\csc(412^\circ) \approx 1.269$

15.  $\cot(-23.1^\circ) \approx -2.344$

16.  $\sec\left(\frac{\pi}{18}\right) \approx 1.015$

17.  $\csc\left(-\frac{3\pi}{4}\right) \approx -1.414$

18.  $\cot(2.61) \approx -1.701$

19.  $x \approx 1.452$  ft.

20.  $x \approx 6.180$  in.

21.  $x \approx 11.706$  ft.

22.  $x \approx 5.016$  in.

23.  $x = 4$  ft.

24.  $x \approx 14.142$  ft.

25.  $x \approx 140.042$  ft.

26.  $x \approx 12.361$  ft.

27.  $x \approx 20.840$  m

### ► Lesson 13

1. 23.8 ft.

2. 39.5 ft.

3. 7.8 ft.

4. 39.9 ft.

## ANSWER KEY

**5.** 23 ft.

**6.** 157.8 ft.

**7.** 1,143 ft.

**8.** 77,274 ft.

**9.** 28 ft.

**10.** 0.88 mi.

**11.** 16.8 ft. to the left and to the right, for a total of 33.6 ft.

**12.** 62.8 ft.

**13.** 39,162 ft.

**14.** 56 ft.

**15.** 18.7 ft.

**16.** 42.9 ft.

**17.** 7,464 ft.

**18.** 1.6 ft.

**19.** 14.6 ft.

**20.** 0.7 mi.

**21.** 30 ft.

**22.** 213 ft.

**23.** 16.2 ft.



24. 7.1 ft.

25. 188.7 ft.

► Lesson 14

1.  $60^\circ = \frac{\pi}{3}$

2.  $60^\circ = \frac{\pi}{3}$

3.  $45^\circ = \frac{\pi}{4}$

4.  $180^\circ = \pi$

5.  $-30^\circ = -\frac{\pi}{6}$

6.  $0^\circ = 0$  radians

7.  $-45^\circ = -\frac{\pi}{4}$

8.  $120^\circ = \frac{2\pi}{3}$

9.  $-60^\circ = -\frac{\pi}{3}$

10.  $0^\circ = 0$

11.  $\frac{5}{\sqrt{29}} = \frac{5\sqrt{29}}{29}$

12.  $\frac{\sqrt{7}}{4}$

13.  $4\sqrt{3}$

14.  $\frac{1}{4}$

15.  $\frac{4}{5}$

16.  $\frac{3}{\sqrt{91}} = \frac{3\sqrt{91}}{91}$

17. 9.2

18.  $\frac{6}{\sqrt{61}} = \frac{6\sqrt{61}}{61}$

19.  $\sin^{-1}(0.81) \approx 54^\circ \approx 0.944$  radians

20.  $\cos^{-1}(-0.33) \approx 109.3^\circ \approx 1.907$  radians

21.  $\tan^{-1}(4.2) \approx 76.6^\circ \approx 1.337$  radians

22.  $\sin^{-1}\left(-\frac{3}{5}\right) \approx -36.87^\circ \approx -0.644$  radians

23.  $\cos^{-1}\left(\frac{11}{16}\right) \approx 46.57^\circ \approx 0.813$  radians

24.  $\tan^{-1}(-1.8) \approx -60.95^\circ \approx -1.064$  radians

25.  $x \approx 23.58^\circ \approx 0.412$  radians

26.  $x \approx 56.25^\circ \approx 0.982$  radians

27.  $x \approx 15.26^\circ \approx 0.266$  radians

## ► Lesson 15

1.  $20^\circ$  west of north

2.  $16.7^\circ$

3.  $76^\circ$  in both directions, for a full span of  $152^\circ$

4.  $8.53^\circ$

5.  $35^\circ$

6.  $1.718^\circ$

7.  $5.43^\circ$  down

**ANSWER KEY**

- 8.**  $21.8^\circ$
- 9.**  $12.37^\circ$
- 10.**  $36.87^\circ$  up
- 11.**  $29.74^\circ$  below the horizontal
- 12.**  $16.26^\circ$
- 13.**  $81.37^\circ$
- 14.**  $40.6^\circ$  up from the horizontal
- 15.**  $43.15^\circ$
- 16.**  $0.48^\circ$
- 17.**  $66.37^\circ$  and  $23.63^\circ$
- 18.**  $51.34^\circ$  and  $38.66^\circ$
- 19.**  $23^\circ$
- 20.**  $27.04^\circ$
- 21.**  $11.3^\circ$
- 22.**  $123.75^\circ$
- 23.**  $81^\circ$ , which remains when  $50^\circ$  and  $49^\circ$  are subtracted from  $180^\circ$
- 24.**  $9.35^\circ$
- 25.**  $10.48^\circ$

**► Lesson 16**

**1.**  $(13.36, 68.71)$

**2.**  $(18.47, -23.64)$

**3.**  $(-66, -17.9)$

**4.**  $(-9.85, 1.74)$

**5.**  $(-50.53, -131.63)$

**6.** length 12.37 at  $14^\circ$  south of west or  $194^\circ$

**7.** length 7.28 at  $74^\circ$  or  $16^\circ$  east of north

**8.** length 24.7 at  $21.4^\circ$  south of east or  $338.6^\circ$

**9.** length 239.6 at  $29.8^\circ$  north of west or  $150.2^\circ$

**10.** length 3.709 at  $31.3^\circ$  east of south or  $301.3^\circ$

**11.**  $150 \cdot \cos(20^\circ) \approx 141$  lbs.

**12.**  $80 \cdot \cos(5^\circ) \approx 79.7$  lbs.

**13.**  $200 \cdot \cos(60^\circ) \approx 100$  lbs.

**14.**  $180 \cdot \cos(37^\circ) \approx 144$  lbs.

**15.**  $130 \cdot \cos(8.6^\circ) \approx 128.5$  lbs.

**16.**  $100 \cdot \cos(75^\circ) \approx 25.9$  lbs.

**17.**  $200 \cdot \cos(72^\circ) \approx 61.8$  lbs.

**18.**  $40 \cdot \cos(48^\circ) \approx 26.8$  lbs.

**19.**  $1,000 \cdot \cos(85^\circ) \approx 87$  lbs.

20.  $100 \cdot \cos(3^\circ) \approx 99.9$  lbs.
21. 68.6 meters from the starting point, heading about  $27^\circ$  west of north
22. about 35.9 miles
23. She is about  $10^\circ$  south of west from where she started, so she should head  $10^\circ$  north of east to go back.
24. The car is about 40 miles from where it started, in the direction that is  $8.5^\circ$  north of east.
25. It is being pulled with about 293 pounds of force in a direction about  $10^\circ$  from the person pulling with 180 pounds of force.

### ► Lesson 17

1.  $x = \frac{14 \cdot \sin(40^\circ)}{\sin(80^\circ)} \approx 9.1$  in.
2.  $x = \frac{5 \cdot \sin(50^\circ)}{\sin(33^\circ)} \approx 7$  ft.
3.  $x = \frac{25 \cdot \sin(140^\circ)}{\sin(25^\circ)} \approx 38$  m
4.  $x = \frac{16 \cdot \sin(75^\circ)}{\sin(25^\circ)} \approx 36.6$  cm
5.  $x = \frac{20 \cdot \sin(88^\circ)}{\sin(41^\circ)} \approx 30.5$  ft.
6.  $x = \frac{90 \cdot \sin(71^\circ)}{\sin(73^\circ)} \approx 89$  in.
7.  $x = \frac{60 \cdot \sin(70^\circ)}{\sin(50^\circ)} \approx 73.6$  m
8.  $x = \frac{100 \cdot \sin(23^\circ)}{\sin(47^\circ)} \approx 53.4$  ft.
9.  $x = \frac{150 \cdot \sin(17^\circ)}{\sin(93^\circ)} \approx 43.9$  in.

$$10. x = \frac{17 \cdot \sin(46^\circ)}{\sin(84^\circ)} \approx 12.3 \text{ ft.}$$

$$11. x = \frac{6 \cdot \sin(80^\circ)}{\sin(50^\circ)} \approx 7.7 \text{ ft. Note: The two unknown angles are both } 50^\circ \text{ because the triangle is isosceles.}$$

$$12. x = \frac{10 \cdot \sin(70^\circ)}{\sin(40^\circ)} \approx 14.6 \text{ ft.}$$

$$13. x \approx 55.5^\circ$$

$$14. x \approx 124.5^\circ$$

$$15. x \approx 74.6^\circ$$

$$16. x \approx 105.4^\circ$$

$$17. x \approx 121.8^\circ$$

$$18. x \approx 58.2^\circ$$

$$19. x \approx 55.2^\circ$$

$$20. x \approx 129.7^\circ$$

$$21. x \approx 29.6^\circ$$

$$22. x \approx 29.4^\circ$$

$$23. x \approx 135.3^\circ$$

## ► Lesson 18

$$1. x = \sqrt{8^2 + 13^2 - 2(8)(13)\cos(58^\circ)} \approx 11.1 \text{ ft.}$$

$$2. x = \sqrt{80^2 + 53^2 - 2(80)(53)\cos(24^\circ)} \approx 38.2 \text{ cm}$$

$$3. x = \sqrt{20^2 + 21^2 - 2(20)(21)\cos(111^\circ)} \approx 33.8 \text{ in.}$$

$$4. x = \sqrt{6^2 + (7.9)^2 - 2(6)(7.9)\cos(71^\circ)} \approx 8.2 \text{ m}$$

5.  $x = \sqrt{5^2 + 12^2 - 2(5)(12)\cos(62^\circ)} \approx 10.6$  ft.
6.  $x = \sqrt{52^2 + 30^2 - 2(52)(30)\cos(81^\circ)} \approx 55.8$  m
7.  $x = \sqrt{16^2 + 18^2 - 2(16)(18)\cos(65^\circ)} \approx 18.3$  cm
8.  $x = \sqrt{11^2 + 9^2 - 2(11)(9)\cos(44^\circ)} \approx 7.7$  ft.
9.  $x = \sqrt{(31.2)^2 + (12.6)^2 - 2(31.2)(12.6)\cos(108^\circ)} \approx 37.1$  m
10.  $x = \sqrt{44^2 + 39^2 - 2(44)(39)\cos(114^\circ)} \approx 69.7$  in.
11.  $x = \sqrt{(9.2)^2 + (12.1)^2 - 2(9.2)(12.1)\cos(88.2^\circ)} \approx 15$  in.
12.  $x = \sqrt{212^2 + 241^2 - 2(212)(241)\cos(29^\circ)} \approx 116.8$  ft.
13.  $x = \cos^{-1}(0.875) \approx 29^\circ$
14.  $x = \cos^{-1}(0.6875) \approx 46.6^\circ$
15.  $x \approx \cos^{-1}(-0.3929) \approx 113.1^\circ$
16.  $x = \cos^{-1}(0.859375) \approx 30.8^\circ$
17.  $x \approx \cos^{-1}(0.3423) \approx 70^\circ$
18.  $x \approx \cos^{-1}(0.9722) \approx 13.5^\circ$
19.  $x = \cos^{-1}(0.859375) \approx 30.8^\circ$
20.  $x \approx \cos^{-1}(0.5323) \approx 57.8^\circ$
21.  $x \approx \cos^{-1}(0.553) \approx 56.4^\circ$
22.  $x \approx \cos^{-1}(0.2174) \approx 77.4^\circ$
23.  $x \approx \cos^{-1}(0.6597) \approx 48.7^\circ$
24.  $x \approx \cos^{-1}(-0.2381) \approx 103.8^\circ$

25.  $x \approx 14.67$  centimeters or  $x \approx 31.29$  cm

26.  $x \approx 10.63$  feet or  $x \approx 19.47$  ft.

27.  $x \approx 21.76$  centimeters or  $x \approx 48.53$  cm

## ► Lesson 19

1. area =  $\sqrt{24} \approx 4.899$  sq. in.

2. area =  $\sqrt{360} \approx 18.974$  sq. ft.

3. area =  $\sqrt{720} \approx 26.833$  sq. mi.

4. area  $\approx 2.905$  sq. in.

5. area  $\approx 8.182$  sq. ft.

6. area  $\approx 18.62$  sq. mi.

7. area =  $\sqrt{896} \approx 29.933$

8. area =  $\sqrt{1,400} \approx 37.417$

9. area =  $\sqrt{48} \approx 6.928$

10. area =  $\sqrt{4,224} \approx 64.992$

11. area  $\approx 23.525$

12. area =  $\sqrt{7,700} \approx 87.750$

13. area  $\approx 585.469$

14. area  $\approx 79.990$

15. area = 726

16. area  $\approx 17.662$



- 17. area  $\approx 12.527$
- 18. area  $\approx 58.557$
- 19. area  $\approx 579.944$
- 20. area  $\approx 143.29$
- 21. area  $\approx 30$  sq. in.

## ► Lesson 20

- 1. The tops of the pistons are about 10.7 inches apart.
- 2. 13.2 cubic feet of mulch will be needed. Remember that 4 inches is  $\frac{1}{3}$  of a foot.
- 3. The smallest angle is the one opposite the 5-inch side, measuring about  $44.4^\circ$ .
- 4. The building is about 160 feet tall.
- 5. The mountain is about 48.3 miles away from the car.
- 6. She is 30.4 feet from where she started.
- 7. The ship is about 45.2 miles from the lighthouse.
- 8. The steeple is about 33.45 feet tall.
- 9. The tips of the antenna are about 16.6 inches apart.
- 10. They should set about  $29.7^\circ$  apart.
- 11. about 232.6 square inches
- 12. The surface area is about 74.5 square millimeters.
- 13. The mountain is about 6.2 miles away.
- 14. They are about 108.5 feet apart.

## ANSWER KEY

- 15.** The area of the plot is about 2 acres.
- 16.** He is about 52.3 feet from the building.
- 17.** The mountain is about 5,395 feet from the first point on the beach.
- 18.** The railing is about 9.1 feet tall.
- 19.** 43.5 square feet of sailcloth are needed. The obtuse angle will be about  $115^\circ$ .
- 20.** She needs about 598 square inches of gold leaf.
- 21.** The second triangle has the largest angle—an obtuse angle of about  $117.3^\circ$ .

# Glossary



**adjacent side** one of the two sides of a triangle that forms the angle in question

**altitude** another name for height

**angle** formed when two straight lines meet at a point

**asymptote** where a graph begins to look like a straight line

**circumference** the distance around a circle, equal to  $C = 2\pi r$ , if the circle has radius  $r$

**complements** angles that add up to  $90^\circ$ , or  $\frac{\pi}{2}$  radians

**components** vectors in the horizontal and vertical directions that combine to form the vector in question

**cosecant** a trigonometric function, written  $\csc(x)$

**cosine** a trigonometric function, written  $\cos(x)$

**cotangent** a trigonometric function, written  $\cot(x)$

**degree** an angle measure equal to  $\frac{1}{360}$  of a full circle

**diagonal** a line that is neither vertical nor horizontal

**domain** the set of all numbers that can be plugged into a function

**equilateral triangle** a triangle with all three sides of the same length

**function** an equation or process that outputs a number for every number plugged into it. We write

$f(x) = y$  to represent a function named  $f$ , which outputs  $y$  when  $x$  is plugged in.

**Heron's formula** the area of a triangle with sides  $A$ ,  $B$ , and  $C$  is  $\sqrt{S(S - A)(S - B)(S - C)}$ , where

$$S = \frac{1}{2}(A + B + C)$$

**hypotenuse** the longest side of a right triangle

**infinite** bigger than any real number

**integers** all the whole numbers, plus their negatives and zero, thus ... -3, -2, -1, 0, 1, 2, 3 ...

**inverse cosine** the function that undoes what cosine does, written either  $\cos^{-1}$  or  $\arccos$ . If  $\cos(x) = y$ , then  $\cos^{-1}(y) = x$ .

**inverse sine** the function that undoes what sine does, written either  $\sin^{-1}$  or  $\arcsin$ . Thus, if  $\sin(x) = y$ , then  $\sin^{-1}(y) = x$ .

**irrational** a number that cannot be written as a fraction of integers

**isosceles triangle** a triangle with two sides of the same length

**Law of Cosines**  $C^2 = A^2 + B^2 - 2AB \cdot \cos(\theta)$ , where  $A$ ,  $B$ , and  $C$  are the sides of a triangle and  $\theta$  is the angle between sides  $A$  and  $B$

**Law of Sines**  $\frac{\sin(a)}{A} = \frac{\sin(b)}{B} = \frac{\sin(c)}{C}$ , where  $A$ ,  $B$ , and  $C$  are the sides of a triangle and  $a$ ,  $b$ , and  $c$  are the angles opposite sides  $A$ ,  $B$ , and  $C$ , respectively

**leg** one of the two shorter sides of a right triangle

**line** a straight one-dimensional object, like a tight string, that extends indefinitely in both directions

**line segment** the piece of a straight line between two endpoints

**magnitude** the length of a vector, usually representing a distance or a force

**minute**  $\frac{1}{60}$  of a degree

**nice angles** multiples of  $30^\circ$ ,  $45^\circ$ ,  $60^\circ$ , and  $90^\circ$   $\left(\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \text{ and } \frac{\pi}{2}\right)$

**opposite angle** the angle of a triangle not formed by the side in question

**opposite side** the side of a triangle that does not form part of the angle in question

**parallel** two lines in a plane that do not intersect

**periodic** a function that repeats at regular intervals

**pi** the ratio of the circumference to the diameter of a circle, written  $\pi = \frac{C}{D}$

**pivot** to move a line segment while keeping one endpoint fixed in place

**plane** a flat, two-dimensional object, like an infinitely big piece of paper

**point** a zero-dimensional object; a single dot in space

**proof** a convincing argument for why something must be true using only facts previously proven or specifically assumed

**Pythagorean theorem**  $A^2 + B^2 = C^2$ , when  $C$  is the hypotenuse of a right triangle with legs  $A$  and  $B$

**Pythagorean triple** a set of three whole numbers  $a$ - $b$ - $c$  with  $a^2 + b^2 = c^2$

## GLOSSARY

**quadratic formula** if  $ax^2 + bx + c = 0$ , then  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

**radian** an angle measure given by the distance traveled around the unit circle. A full circle is  $2\pi$  radians.

**radius** the distance from the center of a circle to any point on the circle

**rational** a number that can be written as a fraction  $\frac{a}{b}$  of integers

**reciprocal** a fraction flipped upside down, like what  $\frac{b}{a}$  is to  $\frac{a}{b}$

**rotate** to pivot in a plane

**reference angle** the angle between a vector and the nearest axis

**right angle** an angle equal to a quarter of a full circle, measuring  $90^\circ$ , or  $\frac{\pi}{2}$  radians

**right triangle** a triangle with a right angle

**scale** to enlarge or shrink a figure by multiplying the length of each side by some number

**scale factor** the number by which a figure is scaled

**secant** a trigonometric function, written  $\sec(x)$

**second**  $\frac{1}{60}$  of a minute

**similar triangles** triangles with the same three angles

**sine** a trigonometric function written  $\sin(x)$

**square root** a symbol that represents the number with a given square. For example,  $\sqrt{2}$  represents the number that, when squared, equals 2.

**squiggle equals** approximately equal to, written  $\approx$

**straight angle** an angle equal to a half of a full circle, measuring  $180^\circ$  or  $\pi$  radians

**tangent** a trigonometric function written  $\tan(x)$

**triangle** a closed figure formed by three sides and three angles

**trigonometric functions**  $\sin(x) = \frac{O}{H}$ ,  $\cos(x) = \frac{A}{H}$ ,  $\tan(x) = \frac{O}{A}$ ,  $\cot(x) = \frac{A}{O}$ ,  $\sec(x) = \frac{H}{A}$ , and  $\csc(x) = \frac{H}{O}$ , where  $H$  is the hypotenuse of a right triangle with angle  $x$ ,  $O$  is the side opposite  $x$ , and  $A$  is the side adjacent to  $x$

**trigonometry** the study of triangles

**unit** a standard term for measuring something like length, force, or time. For example, inches, pounds, and hours are all units.

**unit circle** a circle with a radius of one unit

**vector** a length with a direction

**vertex** the point where two lines meet and form an angle

